Peter Smith, *Introduction to Formal Logic* (CUP, 2nd edition)

These exercises cover the Fitch-style natural deduction system for propositional logic, as used in IFL2.

The key thing for philosophy students is to understand *the principles behind a natural deduction system (and ideally, to get some sense of what are the deep ideas, and what are matters of presentational choice). Doing some exercises no doubt helps understanding; but getting expert at proof-discovery is a different accomplishment. Still, I have provided extensive worked answers with a lot of hints for proof discovery – 44 pages in fact! Find them at this webpage.*

Exercises 20

This set of exercises assumes knowledge of the conjunction and negation rules – i.e. (∧I) and (∧E), (Abs), (RAA) and (DN) – together with (EFQ). See the diagrammatic summary of the rules).

The starred exercises here and in the next two sets are metalogical extras (later used in a completeness proof); though not particularly taxing, they can be skipped.

(a) Show that the following inferences (in suitable languages, of course) can be warranted by proofs using our rules for conjunction and negation:

(1) $\mathsf{P},\ \mathsf{Q},\ \mathsf{R}\ \therefore\ (\mathsf{P} \land (\mathsf{P} \land (\mathsf{Q} \land \mathsf{Q})))$

(2) $(\mathsf{P} \land (\mathsf{Q} \land \mathsf{R}))\ \therefore\ ((\mathsf{Q} \land \neg\neg\mathsf{P}) \land \mathsf{R})$

(3) $(\mathsf{P} \land \mathsf{Q}),\ \neg(\mathsf{P} \land \mathsf{R}),\ \neg(\mathsf{Q} \land \mathsf{S})\ \therefore\ (\neg\mathsf{R} \land \neg\mathsf{S})$

(4) $\neg(\mathsf{P} \land \neg\mathsf{Q}),\ \neg(\mathsf{Q} \land \neg\neg\mathsf{R})\ \therefore\ \neg(\mathsf{P} \land \neg\neg\mathsf{R})$

(5) $\neg((\mathsf{P} \lor \mathsf{R}) \land \neg\neg(\neg\mathsf{S} \land \mathsf{Q})),\ \neg\neg(\neg\mathsf{S} \land \mathsf{Q})\ \therefore\ \neg(\mathsf{P} \lor \mathsf{R})$

(6) $\neg(\mathsf{P} \land \mathsf{S}),\ \neg(\neg\mathsf{S} \land \mathsf{Q})\ \therefore\ \neg((\mathsf{P} \land \mathsf{R}) \land \mathsf{Q})$

(7) $\neg(\mathsf{P} \land \neg(\mathsf{S} \land \mathsf{Q})),\ (\neg\mathsf{R} \land \neg\neg\mathsf{P})\ \therefore\ (\mathsf{Q} \land \neg\neg\neg\mathsf{R})$

(8) $\neg(\mathsf{P} \land \mathsf{S}),\ \neg(\neg\mathsf{S} \land \mathsf{Q}),\ ((\mathsf{P} \land \mathsf{R}) \land \mathsf{Q})\ \therefore\ \mathsf{P}'$

(9) $\neg(\mathsf{P} \land \neg\neg\neg\bot)\ \therefore\ \neg\mathsf{P}$

(10) $(\mathsf{P} \to \mathsf{Q})\ \therefore ((\mathsf{P} \to \mathsf{Q})\ \land \neg\bot)$

(b*) Recall the 'Γ' notation from Exercises 16(c*), introduced to indicate some wffs (zero, one, or many), with 'Γ, α' indicating those wffs together with α. And recall the use of 'iff' introduced in §18.6. We now add a new pair of definitions

> Γ are *S-consistent* – i.e., are consistent as far as the proof system S can tell – iff there is no proof in system S of $\bot$ from Γ as premisses.
>
> Γ are *S-inconsistent* iff there is an S-proof of $\bot$ from Γ as premisses.

Let S be the current proof system with our conjunction and negation rules. Show:

(1) α can be derived in S from Γ as premisses iff $\Gamma, \neg\alpha$ are S-inconsistent.

Now for three results (for eventual use in the Appendix) about what we can *add* to S-consistent wffs while keeping them S-consistent. First, note that if Γ, α are S-inconsistent, Γ proves $\neg\alpha$; so if Γ, α are S-inconsistent and $\neg\neg\alpha$ is one of the wffs Γ, then Γ must already be S-inconsistent. (Explain why!) Conclude that

(2) If the wffs Γ are S-consistent and $\neg\neg\alpha$ is one of them, then Γ, α are also S-consistent.

We use 'Γ, α, β' to indicate the wffs Γ together with α and β. Show that

(3) If the wffs Γ are S-consistent and $(\alpha \wedge \beta)$ is one of them, then Γ, α, β are also S-consistent.

Note too that if $\Gamma, \neg\alpha$ and $\Gamma, \neg\beta$ are both S-inconsistent, we can derive both α and β from Γ, and hence can derive $(\alpha \wedge \beta)$. So if $\Gamma, \neg\alpha$ and $\Gamma, \neg\beta$ are both S-inconsistent and these wffs Γ already include $\neg(\alpha \wedge \beta)$, then Γ are S-inconsistent (why?). Conclude

(4) If the wffs Γ are S-consistent and $\neg(\alpha \wedge \beta)$ is one of them, then either $\Gamma, \neg\alpha$ or $\Gamma, \neg\beta$ (or both) are also S-consistent.

Exercises 21

This set of exercises assumes knowledge of the conjunction and negation rules and now also the rule (Iter) and the two disjunction rules. Again see the diagrammatic summary of the rules).

(a) Show that the following inferences can be warranted by proofs using the derivation rules introduced over the last two chapters (i.e. the rules for negation, conjunction and disjunction, as listed in this summary):

(1) $(\mathsf{P} \vee (\mathsf{Q} \wedge \mathsf{R})) \;\therefore\; ((\mathsf{P} \vee \mathsf{Q}) \wedge (\mathsf{P} \vee \mathsf{R}))$

(2) $(\mathsf{P} \vee \mathsf{Q}) \;\therefore\; \neg(\neg\mathsf{P} \wedge \neg\mathsf{Q})$

(3) $\neg(\mathsf{P} \wedge \mathsf{Q}) \;\therefore\; (\neg\mathsf{P} \vee \neg\mathsf{Q})$

(4) $(\mathsf{P} \vee \mathsf{Q}),\ (\mathsf{P} \vee \mathsf{R}) \;\therefore\; (\mathsf{P} \vee (\mathsf{Q} \wedge \mathsf{R}))$

(5) $(\mathsf{P} \vee \bot),\ (\mathsf{Q} \vee \bot) \;\therefore\; (\mathsf{P} \wedge \mathsf{Q})$

(6) $\neg(\mathsf{Q} \wedge \mathsf{P}),\ ((\mathsf{R} \wedge \mathsf{Q}) \vee (\mathsf{P} \wedge \mathsf{Q})) \therefore\; \mathsf{R}$

(7) $(\mathsf{P} \vee \neg\mathsf{Q}),\ (\mathsf{R} \vee \neg\mathsf{P}),\ (\neg\neg\mathsf{R} \vee \mathsf{Q}) \;\therefore\; \mathsf{R}.$

(8) $(\mathsf{P} \wedge (\mathsf{Q} \vee \mathsf{R})),\ \neg((\mathsf{P} \wedge \mathsf{Q}) \wedge \mathsf{S}),\ (\neg(\mathsf{P} \wedge \mathsf{S}) \vee \neg\mathsf{R}) \;\therefore\; \neg\mathsf{S}$

(b*) Revisit Exercises 20(b*). Let S be the proof system with our rules for conjunction, negation and now disjunction as well.

(1) Do results (1) to (4) from those previous exercises still obtain now we have revised what counts as the proof system S?

Use similar arguments to those outlined in those previous exercises to show:

(2) If the wffs Γ are S-consistent and $(\alpha \vee \beta)$ is one of those wffs, then either Γ, α or Γ, β (or both) are also S-consistent.

(3) If the wffs Γ are S-consistent and $\neg(\alpha \vee \beta)$ is one of those wffs, then $\Gamma, \neg\alpha, \neg\beta$ are also S-consistent.

Exercises 22

We have now at last added the standard two rules for conditionals to get our full classical Fitch-style system: so we have introduction and elimination rules for the four basic connectives, plus (EFQ); the double negation rule (DN), and for convenience the iteration rule (Iter).

(a) Warrant the following inferences by PL natural deduction proofs:

(1) $((\mathsf{P} \land \mathsf{Q}) \to \mathsf{R}) \;\therefore\; (\mathsf{P} \to (\mathsf{Q} \to \mathsf{R}))$
(2) $(\mathsf{P} \to (\mathsf{Q} \to \mathsf{R})) \;\therefore\; ((\mathsf{P} \land \mathsf{Q}) \to \mathsf{R})$
(3) $((\mathsf{P} \lor (\mathsf{Q} \land \mathsf{R})) \to \bot) \;\therefore\; \neg(\mathsf{P} \lor (\mathsf{Q} \land \mathsf{R}))$
(4) $(\mathsf{P} \to \bot),\ (\mathsf{P} \lor \neg\mathsf{Q}) \;\therefore\; (\mathsf{Q} \to \bot)$
(5) $(\mathsf{P} \land (\neg\mathsf{Q} \to \neg\mathsf{P})) \;\therefore\; (\neg\mathsf{P} \lor (\mathsf{Q} \land \mathsf{P}))$
(6) $((\mathsf{P} \land \mathsf{Q}) \to (\mathsf{Q} \land \mathsf{R})),\ (\mathsf{R} \to \neg\mathsf{P}) \;\therefore\; (\mathsf{P} \to \neg\mathsf{Q})$
(7) $(\neg\mathsf{S} \to \neg\mathsf{R}),\ ((\mathsf{P} \land \mathsf{Q}) \lor \mathsf{R}),\ (\neg\mathsf{S} \to \neg\mathsf{Q}) \;\therefore\; (\neg\mathsf{P} \lor \mathsf{S})$
(8) $(\neg\mathsf{P} \to (\mathsf{Q} \land \mathsf{R})),\ \neg(\mathsf{R} \lor \mathsf{P}) \;\therefore\; \neg\mathsf{Q}$

Also give proofs warranting the following inferences:

(9) $\mathsf{Q} \;\therefore\; (\mathsf{P} \to \mathsf{Q})$
(10) $\neg(\mathsf{P} \to \mathsf{Q}) \;\therefore\; \mathsf{P}$
(11) $\neg(\mathsf{P} \to \mathsf{Q}) \;\therefore\; \neg\mathsf{Q}$
(12) $(\mathsf{P} \to (\mathsf{Q} \lor \mathsf{R})) \;\therefore\; ((\mathsf{P} \to \mathsf{Q}) \lor (\mathsf{P} \to \mathsf{R}))$

(b) Following the general definition in Exercises 20(b*), let's say in particular

> Some wffs are PL-consistent if we cannot use premisses from among them to prove $\bot$.

In each of the following cases, show that the given wffs are PL-*in*consistent, i.e. show that there *is* a PL proof of absurdity from them as premisses:

(1) $(\mathsf{P} \to \neg\mathsf{P}),\ (\neg\mathsf{P} \to \mathsf{P})$
(2) $(\neg\mathsf{P} \lor \neg\mathsf{Q}),\ (\mathsf{P} \land \mathsf{Q})$
(3) $((\mathsf{P} \to \mathsf{Q}) \land (\mathsf{Q} \to \neg\mathsf{P})),\ (\mathsf{R} \to \mathsf{P}),\ (\neg\mathsf{R} \to \mathsf{P})$
(4) $(\mathsf{P} \lor (\mathsf{Q} \to \mathsf{R})),\ (\neg\mathsf{R} \land \neg(\mathsf{P} \lor \neg\mathsf{Q}))$
(5) $(\neg\mathsf{P} \lor \mathsf{R}),\ \neg(\mathsf{R} \lor \mathsf{S}),\ (\mathsf{P} \lor \mathsf{Q}),\ \neg(\mathsf{Q} \land \neg\mathsf{S})$.

(c) Suppose that we use '$\leftrightarrow$' so that an expression of the form $(\alpha \leftrightarrow \gamma)$ is simply an *abbreviation* of the corresponding expression of the form $((\alpha \to \gamma) \land (\gamma \to \alpha))$. Warrant the following inferences by PL natural deduction proofs:

(1) $(\mathsf{P} \leftrightarrow \mathsf{Q}) \;\therefore\; (\mathsf{Q} \leftrightarrow \mathsf{P})$
(2) $(\mathsf{P} \leftrightarrow \mathsf{Q}) \;\therefore\; (\neg\mathsf{P} \leftrightarrow \neg\mathsf{Q})$
(3) $(\mathsf{P} \leftrightarrow \mathsf{Q}),\ (\mathsf{Q} \leftrightarrow \mathsf{R}) \therefore\; (\mathsf{P} \leftrightarrow \mathsf{R})$

Suppose alternatively that we introduce '$\leftrightarrow$' to PL as a basic built-in biconditional connective. Give introduction and elimination rules for this new connective (rules which, like the rules for '$\to$', *don't* mention any other connective). Use these new rules to warrant (1) to (3) again. Also give proofs to warrant the following:

(4) $\mathsf{P},\ \mathsf{Q} \;\therefore\; (\mathsf{P} \leftrightarrow \mathsf{Q})$
(5) $\neg(\mathsf{P} \leftrightarrow \mathsf{Q}) \;\therefore\; ((\mathsf{P} \land \neg\mathsf{Q}) \lor (\neg\mathsf{P} \land \mathsf{Q}))$

(6) $(\mathsf{P} \leftrightarrow \mathsf{R}),\ (\mathsf{Q} \leftrightarrow \mathsf{S}) \therefore\ ((\mathsf{P} \vee \mathsf{Q}) \leftrightarrow (\mathsf{R} \vee \mathsf{S}))$

Use a truth-table to confirm that the following wffs are tautologically equivalent:

(7) $(\mathsf{P} \leftrightarrow (\mathsf{Q} \leftrightarrow \mathsf{R})),\ ((\mathsf{P} \leftrightarrow \mathsf{Q}) \leftrightarrow \mathsf{R})$.

For a trickier challenge, outline a proof deriving the second from the first.

(d*) First show

(1) There is a proof of $(\alpha \to \gamma)$ from the premisses Γ if and only if there is a proof of γ from Γ, α.
(2) There is a proof of $(\gamma \to \perp)$ from the premisses Γ if and only if there is a proof of $\neg\gamma$ from Γ.

And now show the following:

(3) The results of Exercises 21(b*) and 22(b*) still obtain when S is the whole PL proof system.
(4) If Γ are PL-consistent and $(\alpha \to \gamma)$ is one of those wffs, then either $\Gamma, \neg\alpha$ or Γ, γ (or both) are also PL-consistent.
(5) If Γ are PL-consistent and $\neg(\alpha \to \gamma)$ is one of those wffs, then $\Gamma, \alpha, \neg\gamma$ are also PL-consistent.

Exercises 23

The starred exercise this time is worth tackling: part of it isn't obvious!

(a) Show that the following wffs are theorems of our PL proof system (in a suitable language).

(1) $(\neg((\mathsf{P} \wedge \mathsf{Q}) \to \mathsf{R}) \vee ((\mathsf{P} \wedge \mathsf{Q}) \to \mathsf{R}))$

(2) $(((\mathsf{P} \wedge \mathsf{Q}) \to \mathsf{R}) \to (\mathsf{Q} \to (\mathsf{P} \to \mathsf{R})))$

(3) $((\mathsf{P} \to (\mathsf{Q} \to \mathsf{R})) \to ((\mathsf{P} \to \mathsf{Q}) \to (\mathsf{P} \to \mathsf{R})))$

(4) $(\neg(\mathsf{P} \wedge (\neg\mathsf{P} \vee \mathsf{Q})) \vee \mathsf{Q})$

(5) $((\mathsf{P} \to \mathsf{Q}) \vee (\mathsf{Q} \to \mathsf{P}))$

(6) $(((\mathsf{P} \wedge \mathsf{Q}) \to \mathsf{R}) \to ((\mathsf{P} \to \mathsf{R}) \vee (\mathsf{Q} \to \mathsf{R})))$

(7) $(((\mathsf{P} \to \mathsf{Q}) \to \mathsf{P}) \to ((\mathsf{Q} \to \mathsf{P}) \vee \mathsf{P}))$

(b*) More on negation and alternative rules of inference. The following rule is often called *Classical Reductio*, to be carefully distinguished from our (RAA):

Given a finished subproof starting with the temporary supposition $\neg\alpha$ and concluding $\perp$, we can derive α.

And the following is a form of *Peirce's Law* (analogous to (LEM)):

We can invoke an instance of $((\alpha \to \beta) \to \alpha) \to \alpha)$ at any stage in a proof.

Show that the new proof system which results from our PL proof system by replacing the double negation rule (DN) with either (i) Classical Reductio or (ii) Peirce's Law is equivalent to our current system.