Peter Smith, *Introduction to Formal Logic* (CUP, 2nd edition)

# Exercises 23: PL Theorems

(a) Show that the following wffs are theorems of our $\mathsf{PL}$ proof system (in a suitable language).

(1) $(\neg((\mathsf{P} \land \mathsf{Q}) \to \mathsf{R}) \lor ((\mathsf{P} \land \mathsf{Q}) \to \mathsf{R}))$

(2) $(((\mathsf{P} \land \mathsf{Q}) \to \mathsf{R}) \to (\mathsf{Q} \to (\mathsf{P} \to \mathsf{R})))$

(3) $((\mathsf{P} \to (\mathsf{Q} \to \mathsf{R})) \to ((\mathsf{P} \to \mathsf{Q}) \to (\mathsf{P} \to \mathsf{R})))$

(4) $(\neg(\mathsf{P} \land (\neg\mathsf{P} \lor \mathsf{Q})) \lor \mathsf{Q})$

(5) $((\mathsf{P} \to \mathsf{Q}) \lor (\mathsf{Q} \to \mathsf{P}))$

(6) $(((\mathsf{P} \land \mathsf{Q}) \to \mathsf{R}) \to ((\mathsf{P} \to \mathsf{R}) \lor (\mathsf{Q} \to \mathsf{R})))$

(7) $(((\mathsf{P} \to \mathsf{Q}) \to \mathsf{P}) \to ((\mathsf{Q} \to \mathsf{P}) \lor \mathsf{P}))$

(b*) More on negation and alternative rules of inference. The following rule is often called *Classical Reductio*, to be carefully distinguished from our (RAA):

> Given a finished subproof starting with the temporary supposition $\neg\alpha$ and concluding $\bot$, we can derive $\alpha$.

And the following is a form of *Peirce's Law* (analogous to (LEM)):

> We can invoke an instance of $((\alpha \to \beta) \to \alpha) \to \alpha)$ at any stage in a proof.

Show that the new proof system which results from our $\mathsf{PL}$ proof system by replacing the double negation rule (DN) with either (i) Classical Reductio or (ii) Peirce's Law is equivalent to our current system.