Peter Smith, *Introduction to Formal Logic* (CUP, 2nd edition)

Exercises 12: Truth functions and truth tables

Give truth tables for the following wffs of a PL *language – i.e. calculate the value of the wff for every assignment of values to the atoms. Use the usual shortcuts, i.e. if a conjunct is false, ignore the other conjunct, as you know the conjunction must be false; if a disjunct is true, ignore the other disjunct, as you know the disjunction must be true; etc. (It is horribly easy to make slips in examples like these. Don't worry too much if you make little errors as long as you understand the ideas in play here. And let me know if I've made a slip!)*

(1) $(P \land \neg(P \land Q))$

P	Q	(P	∧	¬	(P ∧ Q))
T	T		$\underline{F}$	F	T
T	F		$\underline{T}$	T	F
F	T		$\underline{F}$		
F	F		$\underline{F}$		
			1/4	3	2

In pedestrian detail: at stage (1) we can immediately fill in the last two lines [conjunction with false first conjunct]. Then we do stages (2) and (3), so at stage (4) we can compete the value of the whole wff of the remaining lines . We've underlined the final result, written under the main connective. Of course you could e.g. have skipped step (2)!

(2) $((R \lor Q) \lor \neg P)$

P	Q	R	((R ∨ Q)	∨	¬P)
T	T	T	T	$\underline{T}$	
T	T	F	T	$\underline{T}$	
T	F	T	T	$\underline{T}$	
T	F	F	F	$\underline{F}$	F
F	T	T		$\underline{T}$	T
F	T	F		$\underline{T}$	T
F	F	T		$\underline{T}$	T
F	F	F		$\underline{T}$	T

We can immediately spot that on all the last four lines, the disjunct $\neg P$ is true so the whole wff is true. And we equally quickly spot that on all the first three lines the disjunct $(R \lor Q)$ is true so the whole wff is true. So we are left with only have one line to deal with!

(3) $(\neg(P \land \neg S) \land \neg\neg R)$

The obvious place to start is with the second conjunct. Working out the value of a double-negated atom is trivial. And when the second conjunct is false, the whole wff is false of course. So we can immediately complete half the table like this:

P	R	S	(¬ (P ∧ ¬S)	∧	¬¬R)
T	T	T			T
T	T	F			T
T	F	T		$\underline{F}$	F
T	F	F		$\underline{F}$	F
F	T	T			T
F	T	F			T
F	F	T		$\underline{F}$	F
F	F	F		$\underline{F}$	F
				2	1

We can then work out the value of the inner conjunction $(\mathsf{P} \land \neg\mathsf{S})$ in our heads and write down its negation. And then we can easily complete the table!

P	R	S	$(\neg(\mathsf{P} \land \neg\mathsf{S})$	$\land$	$\neg\neg\mathsf{R})$
T	T	T	T	T	T
T	T	F	F	F	T
T	F	T		F	F
T	F	F		F	F
F	T	T	T	T	T
F	T	F	T	T	T
F	F	T		F	F
F	F	F		F	F
			3	2/4	1

(4) $((\mathsf{P} \land \mathsf{Q}) \lor (\neg\mathsf{P} \lor \neg\mathsf{Q}))$ Evidently, when P and Q are both true (first line of the truth table), the first disjunct $(\mathsf{P} \land \mathsf{Q})$ is true, and so the whole wff is true.

Evidently, when at least one of P and Q is false (the other lines of the truth table), the second disjunct $(\neg\mathsf{P} \lor \neg\mathsf{Q})$ is true, and so the whole wff is true.

So the overall wff is always true, on every line – it's a tautology, in the terminology to be introduced in Chapter 14.

P	Q	$((\mathsf{P} \land \mathsf{Q})$	$\lor$	$(\neg\mathsf{P} \lor \neg\mathsf{Q}))$
T	T	T	T	
T	F		T	T
F	T		T	T
F	F		T	T

(5) $\neg((\mathsf{P} \land \neg\mathsf{Q}) \lor (\neg\mathsf{R} \lor \neg(\mathsf{P} \lor \mathsf{Q})))$

This is more tedious work. But take it in stages:

Steps 1, 2 Work out the value of the simpler, first inner disjunct, $(\mathsf{P} \land \neg\mathsf{Q})$. It is true on just two lines, which then settle the value of the whole disjunction.

Steps 3, 4 Work out the value of the other disjunct, $(\neg\mathsf{R} \lor \neg(\mathsf{P} \lor \mathsf{Q}))$ on the remaining six lines. By now you should be able to do this in your head, and that then will settle the value of the whole disjunction on these six lines.

Step 5 Flip the value of the whole disjunction to get the value of its negation, our original wff!

P	Q	R	$\neg$	$((\mathsf{P} \land \neg\mathsf{Q})$	$\lor$	$(\neg\mathsf{R} \lor \neg(\mathsf{P} \lor \mathsf{Q})))$
T	T	T	T	F	F	F
T	T	F	F	F	T	T
T	F	T	F	T	T	
T	F	F	F	T	T	
F	T	T	T	F	F	F
F	T	F	F	F	T	T
F	F	T	F	F	T	T
F	F	F	F	F	T	T
			5	1	2/4	3

As you can see, this sort of exercise quickly becomes tedious. So although it is extremely important to *understand* how a PL wff's value it is fixed by the values of its atoms, don't get too bogged down by the *practicalities* of working through the details! Still, having set another four examples, I'd better continue giving the worked answers here!

(6) $(((P \lor \neg Q) \land (Q \lor R)) \lor \neg\neg(Q \lor \neg R))$

Overall this wff is a disjunction: let's start by looking at the second disjunct $\neg\neg(Q \lor \neg R)$. This of course takes the same truth value as $(Q \lor \neg R)$. It is trivial to work out the value of this on the first four lines; and then it will take the same values in order on the second four lines (why? because the value only depends on the values of Q and R, which repeat).

On six lines, then, that second disjunct is true, and so the whole wff is true:

P	Q	R	$(((P \lor \neg Q) \land (Q \lor R))$	$\lor$	$\neg\neg(Q \lor \neg R))$
T	T	T		$\underline{\text{T}}$	T
T	T	F		$\underline{\text{T}}$	T
T	F	T			F
T	F	F		$\underline{\text{T}}$	T
F	T	T		$\underline{\text{T}}$	T
F	T	F		$\underline{\text{T}}$	T
F	F	T			F
F	F	F		$\underline{\text{T}}$	T
				2	1

So now we just have to look at the first disjunct of our wff on two lines. That's a conjunction of $(P \lor \neg Q)$ and $(Q \lor R)$ – and both of those are true on each of the two relevant lines, making that first disjunct, and hence the whole disjunction, true:

P	Q	R	$(((P \lor \neg Q)$	$\land$	$(Q \lor R))$	$\lor$	$\neg\neg(Q \lor \neg R))$
T	T	T				$\underline{\text{T}}$	T
T	T	F				$\underline{\text{T}}$	T
T	F	T	T	T	T	$\underline{\text{T}}$	F
T	F	F				$\underline{\text{T}}$	T
F	T	T				$\underline{\text{T}}$	T
F	T	F				$\underline{\text{T}}$	T
F	F	T	T	T	T	$\underline{\text{T}}$	F
F	F	F				$\underline{\text{T}}$	T
			3	5	4	2/6	1

(7) $(\neg(\neg P \lor \neg(Q \land \neg R)) \lor \neg\neg(Q \lor \neg P))$

Ooops, I'm sure I didn't intend an example quite so similar to the last one! (I suspect that at some stage a deleted example was accidentally restored!) Obviously we tackle it in the same way, by starting with the second disjunct $\neg\neg(Q \lor \neg P)$, which of course always takes the same value as $(Q \lor \neg P)$, which we can evaluate in our heads, to give us:

P	Q	R	$(\neg(\neg P \lor \neg(Q \land \neg R))$	$\lor$	$\neg\neg(Q \lor \neg P))$
T	T	T		$\underline{\text{T}}$	T
T	T	F		$\underline{\text{T}}$	T
T	F	T			F
T	F	F			F
F	T	T		$\underline{\text{T}}$	T
F	T	F		$\underline{\text{T}}$	T
F	F	T		$\underline{\text{T}}$	T
F	F	F		$\underline{\text{T}}$	T

And again we are left with only two lines to complete – and you can evaluate $\neg(Q \land \neg R)$ in your head, to give you the value of $(\neg P \lor \neg(Q \land \neg R))$ and hence of its negation $\neg(\neg P \lor \neg(Q \land \neg R))$:

P	Q	R	(¬(¬P	∨	¬(Q ∧ ¬R))	∨	¬¬(Q ∨ ¬P))
T	T	T				$\underline{\mathrm{T}}$	T
T	T	F				$\underline{\mathrm{T}}$	T
T	F	T	F	T	T	$\underline{\mathrm{F}}$	F
T	F	F	F	T	T	$\underline{\mathrm{F}}$	F
F	T	T				$\underline{\mathrm{T}}$	T
F	T	F				$\underline{\mathrm{T}}$	T
F	F	T				$\underline{\mathrm{T}}$	T
F	F	F				$\underline{\mathrm{T}}$	T
			5	4	3	2/6	1

(8) (¬((R ∨ ¬Q) ∧ ¬S) ∧ ¬((¬P ∧ Q) ∧ S))

Let's evaluate the first conjunct.

Step 1 It is easy enough to first work out the value of ((R ∨ ¬Q) ∧ ¬S) in your head on the first *eight* lines, and then ...

Step 2 flip values to get the value of the negated wff ¬((R ∨ ¬Q) ∧ ¬S).

Step 3 Since the wff doesn't involve P, we can just copy those same eight values to the next block of eight lines which have the same pattern of values for the relevant atoms Q, R, S.

Step 4 We now know the value of the first conjunct at each line; when this first conjunct is false, the whole conjunction is false.

So we get this far:

P	Q	R	S	(¬((R ∨ ¬Q)	∧	¬S)	∧	¬((¬P ∧ Q)	∧ S))
T	T	T	T	T	F				
T	T	T	F	F	T		$\underline{\mathrm{F}}$		
T	T	F	T	T	F				
T	T	F	F	T	F				
T	F	T	T	T	F				
T	F	T	F	F	T		$\underline{\mathrm{F}}$		
T	F	F	T	T	F				
T	F	F	F	F	T		$\underline{\mathrm{F}}$		
F	T	T	T	T					
F	T	T	F	F			$\underline{\mathrm{F}}$		
F	T	F	T	T					
F	T	F	F	T					
F	F	T	T	T					
F	F	T	F	F			$\underline{\mathrm{F}}$		
F	F	F	T	T					
F	F	F	F	F			$\underline{\mathrm{F}}$		
				2/3	1		4		

So now moving on:

Step 5 It is easy enough to first work out the value of ((¬P ∧ Q) ∧ S) in your head on the ten lines, we haven't yet settled the final value of. It is only true on the two lines when P is false while Q and S are true! Otherwise it has to be false.

Step 6 Flip values to get the value of the negated wff ¬((¬P ∧ Q) ∧ S) .

Step 7 We can *now* calculate the overall value of the wff on the ten as-yet-unsettled lines.

P	Q	R	S	(¬((R ∨ ¬Q)	∧	¬S)	∧	¬((¬P ∧ Q)	∧ S))
T	T	T	T	T	F		$\underline{\mathrm{T}}$	T	F
T	T	T	F	F	T		$\underline{\mathrm{F}}$		
T	T	F	T	T	F		$\underline{\mathrm{T}}$	T	F
T	T	F	F	T	F		$\underline{\mathrm{T}}$	T	F
T	F	T	T	T	F		$\underline{\mathrm{T}}$	T	F
T	F	T	F	F	T		$\underline{\mathrm{F}}$		
T	F	F	T	T	F		$\underline{\mathrm{T}}$	T	F
T	F	F	F	F	T		$\underline{\mathrm{F}}$		
F	T	T	T	T			$\underline{\mathrm{F}}$	F	T
F	T	T	F	F			$\underline{\mathrm{F}}$		
F	T	F	T	T			$\underline{\mathrm{F}}$	F	T
F	T	F	F	T			$\underline{\mathrm{T}}$	T	F
F	F	T	T	T			$\underline{\mathrm{T}}$	T	F
F	F	T	F	F			$\underline{\mathrm{F}}$		
F	F	F	T	T			$\underline{\mathrm{T}}$	T	F
F	F	F	F	F			$\underline{\mathrm{F}}$		
				2/3	1		4/7	6	5

(9) (¬(¬(P ∧ Q) ∧ ¬(R ∧ S)) ∨ ¬(S ∧ (Q ∨ R)))

Start with the second disjunct:

Step 1 Let's do more in our heads! When is (S ∧ (Q ∨ R)) *false*? Often! – just when S is false or when both Q and R are false. Hence on just those lines its negation ¬(S ∧ (Q ∨ R)) will be *true*. So write down those values on the first eight lines . . .

Step 2 and copy the same values to the next block of eight lines which have the same pattern of values for the relevant atoms Q, R, S.

Step 3 We now know the value of the whole wff, whenever its second disjunct is true:

P	Q	R	S	(¬(¬(P ∧ Q) ∧ ¬(R ∧ S))	∨	¬(S ∧ (Q ∨ R)))
T	T	T	T			F
T	T	T	F		$\underline{\mathrm{T}}$	T
T	T	F	T			F
T	T	F	F		$\underline{\mathrm{T}}$	T
T	F	T	T			F
T	F	T	F		$\underline{\mathrm{T}}$	T
T	F	F	T		$\underline{\mathrm{T}}$	T
T	F	F	F		$\underline{\mathrm{T}}$	T
F	T	T	T			F
F	T	T	F		$\underline{\mathrm{T}}$	T
F	T	F	T			F
F	T	F	F		$\underline{\mathrm{T}}$	T
F	F	T	T			F
F	F	T	F		$\underline{\mathrm{T}}$	T
F	F	F	T		$\underline{\mathrm{T}}$	T
F	F	F	F		$\underline{\mathrm{T}}$	T
					3	1/2

That leaves us the task of evaluating the other disjunct of the original wff on just six lines.

Step 4 Again we can do more in our heads: it isn't hard to evaluate (P ∧ Q) ∧ ¬(R ∧ S)).

Step 5 Flip values to get the value of its negation, i.e. the first disjunct.

Step 6 We can now complete the table.

P	Q	R	S	$(\neg(\neg(P \wedge Q)$	$\wedge$	$\neg(R \wedge S))$	$\vee$	$\neg(S \wedge (Q \vee R)))$
T	T	T	T	T	F		$\underline{\text{T}}$	F
T	T	T	F				$\underline{\text{T}}$	T
T	T	F	T	T	F		$\underline{\text{T}}$	F
T	T	F	F				$\underline{\text{T}}$	T
T	F	T	T	T	F		$\underline{\text{T}}$	F
T	F	T	F				$\underline{\text{T}}$	T
T	F	F	T				$\underline{\text{T}}$	T
T	F	F	F				$\underline{\text{T}}$	T
F	T	T	T	T	F		$\underline{\text{T}}$	F
F	T	T	F				$\underline{\text{T}}$	T
F	T	F	T	F	T		$\underline{\text{F}}$	F
F	T	F	F				$\underline{\text{T}}$	T
F	F	T	T	T	F		$\underline{\text{T}}$	F
F	F	T	F				$\underline{\text{T}}$	T
F	F	F	T				$\underline{\text{T}}$	T
F	F	F	F				$\underline{\text{T}}$	T
				5	4		3/6	1/2

Which wasn't *too* gruesome . . .