Peter Smith, *Introduction to Formal Logic* (CUP, 2nd edition)

Exercises 14: Tautologies

(a) *Which of the following wffs are tautologies, which are contradictions, and which are neither?*

(1) $\neg((\neg\neg Q \land \neg\neg P) \lor \neg(P \land Q))$

Since $(\neg\neg Q \land \neg\neg P)$ is equivalent to $(P \land Q)$, this wff is equivalent to one of the form $\neg(\alpha \lor \neg\alpha)$, which is the negation of a tautology, so is a contradiction.

Or you could answer this the laborious way, and do a truth table!

(2) $(P \lor ((\neg P \lor Q) \land \neg Q))$

By inspection, if P is true, then this wff is true (true first disjunct).

If P is false and Q is true then this wff is false (both disjuncts are false).

So this is neither a tautology or a contradiction.

(You could do a full truth table to get to the same result, but you don't need to!).

(3) $(\{P \lor \neg(Q \land R)\} \lor \{(\neg P \land Q) \land R\})$

Being careless with unnecessary brackets, the first disjunct $P \lor \neg(Q \land R)$ is equivalent to $P \lor \neg Q \lor \neg R$. And the negation of that (by a DeMorgan's Law) is $\neg P \land Q \land R$. So in fact our wff is equivalent to one of the form $(\alpha \lor \neg\alpha)$ and is a tautology.

Or here is a truth table, done with some obvious short cuts:

P	Q	R	({P	∨	¬(Q ∧ R)}	∨	{(¬P ∧ Q)	∧	R})
T	T	T		T		$\underline{\mathrm{T}}$			
T	T	F		T		$\underline{\mathrm{T}}$			
T	F	T		T		$\underline{\mathrm{T}}$			
T	F	F		T		$\underline{\mathrm{T}}$			
F	T	T		F		$\underline{\mathrm{T}}$		T	
F	T	F		T		$\underline{\mathrm{T}}$			
F	F	T		T		$\underline{\mathrm{T}}$			
F	F	F		T		$\underline{\mathrm{T}}$			
				1		2/4		3	

(1) evaluate the left-hand curly-bracketed disjunct. The first four lines are easy, and have to be true since P is true. The disjunct's value on the remaining lines will depend on the value of $\neg(Q \land R)$ which is easily computed. (2) That fixes the overall value of the wff on seven lines(!). (3) Determine the value of the right-hand curly-bracketed disjunct on the one remaining line, and (4) that settles the overall value for the wff on that line.

(4) $(\{P \lor (Q \land \neg R)\} \lor \neg\{(\neg P \lor R) \lor Q\})$

You could do a brute-force truth table. But pause for thought.

Can we make the wff true? When P is true, the first curly bracketed expression is true. So the whole wff is true.

Can we make the wff false? We need to simultaneously make both disjuncts false. Suppose P is false and Q is false. Then (i) the first disjunct is false (because both disjuncts inside the first curly-bracketed expression are false). And (ii) the second curly bracketed expression is true because $\neg P$ is true: hence its negation, the second disjunct of the whole wff, is also false.

So the whole wff is neither a tautology or a contradiction.

(5) $(\{\mathsf{P} \wedge (\neg\mathsf{Q} \vee \neg\mathsf{R})\} \vee \neg\{(\mathsf{P} \vee \neg\mathsf{R}) \vee \neg\mathsf{Q}\})$

You could do a brute-force truth table. But again pausing for thought is better.

Can we make the wff true? Yes. For example, when P is true and Q and R are both false, the first curly bracketed expression is true – so then the whole wff is true.

Can we make the wff false? Yes. We need to simultaneously make both disjuncts false. How can we make first disjunct false? By making P false or by making Q and R both true. How can we make the second disjunct false? By making one of P or $\neg\mathsf{Q}$ and $\neg\mathsf{R}$ true. So if we make P true while Q and R are also true the whole wff is false.

So the whole wff is neither a tautology or a contradiction

(6) $\neg(\{(\mathsf{P} \wedge \neg(\mathsf{Q} \wedge \mathsf{S})) \vee \neg\mathsf{R}\} \wedge \neg\{(\mathsf{P} \wedge \neg(\mathsf{Q} \wedge \mathsf{S})) \vee \neg\mathsf{R}\})$

I hope you quickly spotted that this has the form $\neg(\alpha \wedge \neg\alpha)$ so is a tautology!

(7) $\neg(\{\neg(\mathsf{P} \wedge \neg\mathsf{R}) \wedge \neg(\mathsf{Q} \wedge \neg\mathsf{S})\} \wedge \neg\{\neg(\mathsf{P} \vee \mathsf{Q}) \vee (\mathsf{R} \vee \mathsf{S})\})$

This is a tautology, as a truth table will confirm.

But with a bit of reflection we can see that it *ought* to be a tautology. For a wff of the form $\neg(\alpha \wedge \neg\beta)$ tells that that we don't have α true and β false; in other words, it tells us that if α then β, or given α then we can infer β.

So now look at our wff. It tells us that given {if P then R, and also if Q then S}, then we can infer {either not-$(\mathsf{P} \vee \mathsf{Q})$ or $(\mathsf{R} \vee \mathsf{S})$}. And that's a logical truth. Because we know that it is a logical truth that {either not-$(\mathsf{P} \vee \mathsf{Q})$ or $(\mathsf{P} \vee \mathsf{Q})$}; and in the second case, given we have either P or Q, then the first curly bracketed expression tells us that we must have either R or S.

(b*) *Which of the following claims are true about PL wffs, and why?*

(1) *The conjunction of a contradiction and any another wff is still a contradiction.*

True. A contradiction is false on every relevant valuation. So its conjunction with anything else will also be false on every relevant valuation.

(2) *The conjunction of a tautology and any another wff is still a tautology.*

False. Suppose, e.g., that the other wff is a contradiction! Then the whole will in fact always be false.

(3) *The disjunction of two tautologies is a tautology.*

True! A tautology is true on every relevant valuation, so its disjunction with anything will also be true on every relevant valuation.

(4) *All the tautologies in a PL language express the same truth function as each other.*

No. The tautology $(\mathsf{P} \vee \neg\mathsf{P})$ expresses a truth function of *one* variable which for any input value of that variable outputs the value T; the tautology $((\mathsf{P} \wedge \mathsf{Q}) \vee \neg(\mathsf{P} \wedge \mathsf{Q})$ expresses a truth function of *two* variables which for any input values of those variables outputs the value T.

(5) *Every contradiction in a PL language has the same truth table as a wff of the form $(\alpha \wedge \neg\alpha)$.*

True. Suppose α is a contradiction. Then of course it has the same truth table as $(\alpha \wedge \neg\alpha)$!

If a wff is neither a tautology nor a contradiction, it is said to be contingent. *Which of the following claims are true, and why?*

(6) *The negation of a contingent wff is contingent.*

Trivially, true! A contingent wff is true on some relevant valuations and false on some others: so its negation will be false on some relevant valuations and true on some others.

(7) *The disjunction of two contingent wffs is contingent.*

False. P is contingent; so is $\neg\mathsf{P}$. But their disjunction $(\mathsf{P} \lor \neg\mathsf{P})$ is not contingent.

(8) *The conjunction of two contingent wffs is contingent.*

False. Again, P is contingent; so is $\neg\mathsf{P}$. But their conjunction $(\mathsf{P} \land \neg\mathsf{P})$ is not contingent.

(9) *The disjunction of two contingent wffs is never a contradiction.*

True. A contingent wff is true on some relevant valuations, and its disjunction with any wff will also be true on those same valuations. But if that disjunction is true on some valuations it can't be a contradiction.