Peter Smith, *Introduction to Formal Logic* (CUP, 2nd edition)

Exercises 15: Tautological entailment

(a) *Use the truth table test to determine which of the following arguments are tautologically valid:*

(1) $(\mathsf{P} \wedge \mathsf{S}), \neg(\mathsf{S} \wedge \neg\mathsf{R}) \therefore (\mathsf{R} \vee \neg\mathsf{P})$

Note that there are just three atoms here, namely P, R, S. So we can construct an eight line truth table as follows, using the standard short-cut tricks. We start by evaluating the first premiss, as that's the simplest wff.

P	R	S	$(\mathsf{P} \wedge \mathsf{S})$	$\neg(\mathsf{S} \wedge \neg\mathsf{R})$	$(\mathsf{R} \vee \neg\mathsf{P})$
T	T	T	T	T	T
T	T	F	F		
T	F	T	T	F	
T	F	F	F		
F	T	T	F		
F	T	F	F		
F	F	T	F		
F	F	F	F		

Immediately we find that six lines can't be 'bad' lines, i.e. lines where the premisses are true and the conclusion false, because that first premiss is false on those six lines.

So we need evaluate the second premiss on just two lines (which we've done 'in our heads').

That leaves just the first line, as it happens, as a potentially bad line with true premisses and perhaps a false conclusion. But the conclusion is true here. So there is no line with true premisses and a false conclusion, and the argument is tautologically valid.

Can you see why this is the expected conclusion? Note that $(\mathsf{P} \wedge \mathsf{S})$ of course implies S; and S and $\neg(\mathsf{S} \wedge \neg\mathsf{R})$ together prove R by what should be a familiar pattern of argument. Finally R implies $(\mathsf{R} \vee \neg\mathsf{P})$.

(2) $\mathsf{P}, \neg(\mathsf{P} \wedge \neg\mathsf{Q}), (\neg\mathsf{Q} \vee \mathsf{R}) \therefore \mathsf{R}$

Again three atomic wffs, so an eight line table is called for! Here's one:

P	Q	R	P	$\neg(\mathsf{P} \wedge \neg\mathsf{Q})$	$(\neg\mathsf{Q} \vee \mathsf{R})$	R
T	T	T	T			T
T	T	F	T	T	F	F
T	F	T	T			T
T	F	F	T	F		F
F	T	T	F			
F	T	F	F			
F	F	T	F			
F	F	F	F			

Again, we have first evaluated the trivially simple initial premiss and then (looking at just the lines that are still potentially bad) we evaluate the trivially simple conclusion.

Next we look at the second premiss on the only two remaining lines with so-far-true-premisses and a false conclusion. The fourth line is now ruled out as a possible bad line.

So we only need to evaluate the third premiss on one remaining potentially-bad line of the table, i.e. the second one! But that premiss is false there.

We are done: there are no bad lines, and the inference is valid. Can you see why this too is the expected result?

(3) $(\neg P \lor \neg(Q \lor R)), (Q \lor (P \land R)) \therefore (\neg P \lor Q)$

Another inference involving just three atoms, so only needing an eight line table. Evaluate the simple conclusion first. And lo and behold, six lines are already ruled out as candidate bad lines!

P	Q	R	$(\neg P \lor \neg(Q \lor R))$	$(Q \lor (P \land R))$	$(\neg P \lor Q)$
T	T	T			T
T	T	F			T
T	F	T			F
T	F	F			F
F	T	T			T
F	T	F			T
F	F	T			T
F	F	F			T

So now evaluate the second premiss, as that is the simpler of the two, and we only need to look at two lines! One of these lines is then immediately ruled out as a possible bad line. We then finally evaluate the first premiss, to get

P	Q	R	$(\neg P \lor \neg(Q \lor R))$	$(Q \lor (P \land R))$	$(\neg P \lor Q)$
T	T	T			T
T	T	F			T
T	F	T	F	T	F
T	F	F		F	F
F	T	T			T
F	T	F			T
F	F	T			T
F	F	F			T

So, perhaps a little more surprisingly this time, we have another valid argument.

(4) $(P \lor Q), \neg(Q \land \neg\neg R) \therefore \neg(R \lor P)$

Another eight line table is required. Let's dive straight in:

P	Q	R	$(P \lor Q)$	$\neg(Q \land \neg\neg R)$	$\neg(R \lor P)$
T	T	T	T	F	F
T	T	F	T	T	F
T	F	T	T		F
T	F	F	T		F
F	T	T	T		F
F	T	F	T		T
F	F	T	F		
F	F	F	F		

We evaluate the first premiss first, as that's the simplest wff, and then the conclusion. At that stage, the first five lines are still potentially bad lines with a true premiss and a false conclusion.

So now we turn to the second premiss. On the first line it comes out false. But on the second line it is true, giving us a bad line. And we can stop here. One bad line is bad enough. The argument is tautologically invalid.

(5) $\mathsf{P}, (\mathsf{Q} \lor \mathsf{R}) \therefore ((\mathsf{P} \land \mathsf{Q}) \lor (\mathsf{P} \land \mathsf{R}))$

A moment's reflection should tell you that this ought to be valid (why?). And indeed it is:

P	Q	R	P	(Q ∨ R)	((P ∧ Q) ∨ (P ∧ R))
T	T	T	T	T	T
T	T	F	T	T	T
T	F	T	T	T	T
T	F	F	T	F	
F	T	T	F		
F	T	F	F		
F	F	T	F		
F	F	F	F		

We have evaluated wffs in order of complexity. We can rule out four lines as potential bad lines after the first step; evaluation the second premiss allows us to rule out another line, leaving us with with three potential bad lines (and breaking the pattern of earlier examples where we were eventually left with just one potential bad line to look at). Evaluating our conclusion, we find that, as we wanted, our inference is indeed tautologically valid.

(6) $(\mathsf{P} \lor (\neg\mathsf{P} \land \mathsf{Q})), (\neg\mathsf{P} \lor \mathsf{R}), \neg(\mathsf{Q} \land \mathsf{S}) \therefore \neg(\neg\mathsf{R} \land \mathsf{S})$

Four atomic wffs are in play, so we need a sixteen line truth table (sorry!).

Where shall we start? The conclusion is a negated conjunction. Conjunctions are hard to make true; negated conjunctions are correspondingly easy to make true. So we can easily make the conclusion true. And lines with true conclusions can't be bad. So let's look at the conclusion first, even though it isn't the simplest wff.

Calculate its value on the first block of four lines (easy!). And then note that the next three blocks of four lines must repeat the same pattern of values, since the assignments of values to R and S repeat.

We can quickly complete then our first column of working:

P	Q	R	S	(P ∨ (¬P ∧ Q))	(¬P ∨ R)	¬(Q ∧ S)	¬(¬R ∧ S)
T	T	T	T				T
T	T	T	F				T
T	T	F	T				F
T	T	F	F				T
T	F	T	T				T
T	F	T	F				T
T	F	F	T				F
T	F	F	F				T
F	T	T	T				T
F	T	T	F				T
F	T	F	T				F
F	T	F	F				T
F	F	T	T				T
F	F	T	F				T
F	F	F	T				F
F	F	F	F				T
				4	2	3	1

And now there are only four potential bad lines and things immediately become more manageable!

So let's next evaluate the middle premiss as that is the simplest one: and we get:

P	Q	R	S	$(\mathsf{P} \lor (\neg\mathsf{P} \land \mathsf{Q}))$	$(\neg\mathsf{P} \lor \mathsf{R})$	$\neg(\mathsf{Q} \land \mathsf{S})$	$\neg(\neg\mathsf{R} \land \mathsf{S})$
T	T	T	T				T
T	T	T	F				T
T	T	F	T		F		F
T	T	F	F				T
T	F	T	T				T
T	F	T	F				T
T	F	F	T		F		F
T	F	F	F				T
F	T	T	T				T
F	T	T	F				T
F	T	F	T		T		F
F	T	F	F				T
F	F	T	T				T
F	F	T	F				T
F	F	F	T		T		F
F	F	F	F				T
				4	2	3	1

That eliminates two more potential bad lines – so we need only to look at the third premiss on two lines, and finally the first premiss on the remaining candidate bad line to finish the table like this:

P	Q	R	S	$(\mathsf{P} \lor (\neg\mathsf{P} \land \mathsf{Q}))$	$(\neg\mathsf{P} \lor \mathsf{R})$	$\neg(\mathsf{Q} \land \mathsf{S})$	$\neg(\neg\mathsf{R} \land \mathsf{S})$
T	T	T	T				T
T	T	T	F				T
T	T	F	T		F		F
T	T	F	F				T
T	F	T	T				T
T	F	T	F				T
T	F	F	T		F		F
T	F	F	F				T
F	T	T	T				T
F	T	T	F				T
F	T	F	T		T	F	F
F	T	F	F				T
F	F	T	T				T
F	F	T	F				T
F	F	F	T	F	T	T	F
F	F	F	F				T
				4	2	3	1

And we find that our table has no bad lines, and the inference is valid. Which wasn't too painful!

Remark: Of course, cases can be potentially a lot nastier/messier than these. But if you've successfully coped with these initial examples, you've at least grasped the principle of truth-table a formalized PL argument for tautological validity!

(b) *Evaluate the following arguments*:

(1) *Either Jack went up the hill or Jill did. Either Jack didn't go up the hill or the water got spilt. Hence, either Jill went up the hill or the water got spilt.*

We now turn to putting our apparatus to work in assessing arguments framed in ordinary language.

Suppose we adopt an ad hoc PL language with the following glossary:

P: Jack went up the hill
Q: Jill went up the hill
R: The water got spilt

So the argument can be regimented like this, if we assume to begin with that the disjunctions are inclusive:

$$(\mathsf{P} \lor \mathsf{Q}),\ (\neg\mathsf{P} \lor \mathsf{R})\ \therefore (\mathsf{Q} \lor \mathsf{R})$$

Let's do an easy truth-table:

P	Q	R	$(\mathsf{P} \lor \mathsf{Q})$	$(\neg\mathsf{P} \lor \mathsf{R})$	$(\mathsf{Q} \lor \mathsf{R})$
T	T	T			T
T	T	F			T
T	F	T			T
T	F	F	T	F	F
F	T	T			T
F	T	F			T
F	F	T			T
F	F	F	F		F

Again, we've evaluated the conclusion first. Why? Because it is disjunctive, so easy to make true. And when we've found a conclusion to be true on lots of lines we can henceforth igmore those lines: they can't be bad lines with a true premiss and false conclusion.

Having evaluated the conclusion we are left with just two potentially bad lines. The table can then be quickly completed, showing that our formal version of the argument is tautologically valid, and hence plain valid. Hence the original informal argument is valid too – at least so long as we can construe the disjunctions as inclusive.

What if we construe some or all of the disjunctions as exclusive?

A general observation: the *exclusive* disjunction of (say) P and Q is equivalent to the conjunction of the *inclusive* $(\mathsf{P} \lor \mathsf{Q})$ plus a clause $\neg(\mathsf{P} \land \mathsf{Q})$ ruling out P and Q being true together. In an obvious sense, therefore, an exclusive disjunction is a *stronger* claim than the corresponding inclusive disjunction. This has two implications:

1. If an argument with an inclusive disjunction as a premiss is valid, so is the same argument with the corresponding exclusive disjunction as a premiss instead. (For if there is no way of making the weaker premiss(es) true and conclusion false, there can be no way of making the stronger premiss(es) true and conclusion false.)
2. An argument with an inclusive disjunction as a conclusion may be valid while the same argument with the corresponding stronger exclusive disjunction as the conclusion is invalid. For example, $\mathsf{P} \therefore (\mathsf{P} \lor \mathsf{Q})$ is valid, but $\mathsf{P} \therefore (\mathsf{P} \oplus \mathsf{Q})$ is invalid (where $\oplus$ signifies exclusive disjunction).

Applying these observations to the present case, if we construe one or both the premisses of the argument as exclusive disjunctions, while keeping the conclusion inclusive, the argument remains valid. While the reverse case, where we construe both the premisses of the argument as inclusive disjunctions and the conclusion as exclusive, then the argument is invalid as is witnessed by the following valuation:

P	Q	R	$(P \vee Q)$	$(\neg P \vee R)$	$(Q \oplus R)$
T	T	T	T	T	F

As a further exercise: what happens if we treat all three disjunctions in our original argument as exclusive?

(2) *Either Jack didn't go up the hill or the water got spilt. Why? Because it isn't the case that Jack went up the hill and Jill didn't. Moreover, it isn't the case that Jill went up the hill and the water got spilt.*

Keeping to the same language with the same glossary, and noting that the conclusion is given first, we can render this argument as follows (again starting by treating the disjunction as inclusive):

$$\neg(P \wedge \neg Q),\ \neg(Q \wedge R)\ \therefore\ (\neg P \vee R)$$

We start our truth table by looking at the disjunctive conclusion first: we then have only two potentially bad lines to explore further, and quickly find a line where the premisses are true and conclusion false:

P	Q	R	$\neg(P \wedge \neg Q)$	$\neg(Q \wedge R)$	$(\neg P \vee R)$
T	T	T			T
T	T	F	T	T	F
T	F	T			T
T	F	F	F		F
F	T	T			T
F	T	F			T
F	F	T			T
F	F	F			T

Hence our formal PL language is not tautologically valid. Moreover, the countervaluation – the valuation which makes the premisses true and conclusion false – corresponds to the situation where Jack and Jill both go up the hill but the water doesn't get spilt. That's possible: there is indeed a possible situation in the premisses of our original argument are true and conclusion false – so the inference is indeed not just tautologically invalid but outright invalid.

What if the disjunctive conclusion had been construed exclusively? Well, if the argument's premiss don't entail the conclusion when it is read as making the weaker inclusive claim, they can't entail the conclusion when it is read as making the stronger exclusive claim!

(3) *Either Jill hasn't trained hard or she will win the race. It isn't true that she'll win the race and not be praised. Either Jill has trained hard or she deserves to lose. Hence either Jill will be praised or she deserves to lose.*

We adopt another ad hoc PL language with the following glossary:

P: Jill has trained hard
Q: Jill will win the race
R: Jill will be praised
S: Jill deserves to lose

So the argument can be regimented like this, if we again assume (to begin with) that the disjunctions are inclusive:

$$(\neg P \vee Q),\ \neg(Q \wedge \neg R),\ (P \vee S)\ \therefore\ (R \vee S)$$

Here then is the corresponding truth-table – where we indicate at the foot of columns the order in which we have tackled wffs, starting with the conclusion.

P	Q	R	S	$(\neg P \lor Q)$	$\neg(Q \land \neg R)$	$(P \lor S)$	$(R \lor S)$
T	T	T	T				T
T	T	T	F				T
T	T	F	T				T
T	T	F	F	T	F	T	F
T	F	T	T				T
T	F	T	F				T
T	F	F	T				T
T	F	F	F	F		T	F
F	T	T	T				T
F	T	T	F				T
F	T	F	T				T
F	T	F	F			F	F
F	F	T	T				T
F	F	T	F				T
F	F	F	T				T
F	F	F	F			F	F
				3	4	2	1

So the inference is tautologically valid, and hence valid.

You might think, however, that the disjunctions in our example are naturally to be construed exclusively, so the argument should be regimented like this:

$$(\neg P \oplus Q),\ \neg(Q \land \neg R),\ (P \oplus S)\ \therefore\ (R \oplus S)$$

So how does the argument then stack up? Here's the revised table:

P	Q	R	S	$(\neg P \oplus Q)$	$\neg(Q \land \neg R)$	$(P \oplus S)$	$(R \oplus S)$
T	T	T	T			F	F
T	T	T	F				T
T	T	F	T				T
T	T	F	F	T	F	T	F
T	F	T	T			F	F
T	F	T	F				T
T	F	F	T				T
T	F	F	F	F		T	F
F	T	T	T	F		T	F
F	T	T	F				T
F	T	F	T				T
F	T	F	F			F	F
F	F	T	T	T	T	T	F
F	F	T	F				T
F	F	F	T				T
F	F	F	F			F	F
				3	4	2	1

This time the argument is invalid. And, reading the bad line off the table, consider the situation where Jill hasn't trained hard, and won't win; and yet she will be praised, even though she deserves to lose. That makes the premisses true, but makes the disjunctive conclusion false if that conclusion is indeed read exclusively.

(4) *Veronika and Marek are both violinists. Veronika and Peter are married but aren't both violinists. And likewise, Marek and Jiří aren't both violinists. So neither Jiří nor Peter are violinists.*

This time, for a start, let's cut ourselves some slack by using more memorable letters for atomic wffs than 'P', 'Q', etc. and adopt a PL type language with the glossary:

V: Veronika is a violinist
M: Marek is a violinist
P: Peter is a violinist
J: Jiří is a violinist

Do we need another atom to record that Veronika and Peter are married? Well, yes, if we are to fully translate everything in the given argument. But a moment's reflection shows that that clause does no work at all in the argument. It is in effect an extra premiss that has no relation to anything else; if the argument is valid at all, it must be valid without it.

As for the conclusion of the argument, we can translate it in two equivalent ways, either as the conjunction of $\neg\mathsf{J}$ and $\neg\mathsf{P}$, or else the negation of the *inclusive* disjunction (it has to be inclusive – why?) of J and P.

So we can regiment the working core of our ordinary-language argument like this:

$$(\mathsf{V} \wedge \mathsf{M}),\ \neg(\mathsf{V} \wedge \mathsf{P}),\ \neg(\mathsf{M} \wedge \mathsf{J})\ \therefore\ \neg(\mathsf{J} \vee \mathsf{P})$$

And now we can do a truth-table test, taking the four wffs in turn:

V	M	P	J	$(\mathsf{V} \wedge \mathsf{M})$	$\neg(\mathsf{V} \wedge \mathsf{P})$	$\neg(\mathsf{M} \wedge \mathsf{J})$	$\neg(\mathsf{J} \vee \mathsf{P})$
T	T	T	T	T	F		
T	T	T	F	T	F		
T	T	F	T	T	T	F	
T	T	F	F	T	T	T	T
T	F	T	T	F			
T	F	T	F	F			
T	F	F	T	F			
T	F	F	F	F			
F	T	T	T	F			
F	T	T	F	F			
F	T	F	T	F			
F	T	F	F	F			
F	F	T	T	F			
F	F	T	F	F			
F	F	F	T	F			
F	F	F	F	F			

Already, after evaluating the first premiss, we find that there are only four possible bad lines. And then the rest of the working goes as you would now expect, to show that there are no bad lines on the table, so the formal inference is tautologically valid; hence it is valid. And hence the original ordinary-language argument with its extra unused premiss is also valid.

(5) *Jill is a mother. Either Jack and Jill are both parents, or neither are. Either Jack isn't a parent at all, or he is a proud father. Therefore Jack is a proud father.*

Intuitively, this is a valid inference. Jill's being a mother entails her being a parent. So by the second premiss, it follows that Jack is a parent too. And by the third premiss, therefore, he is a proud father.

But the validity of the inference doesn't just depend on just the meaning of the propositional connectives. If we just baldly regiment the argument using a PL language with the glossary

P: Jill is a mother
Q: Jill is a parent
R: Jack is a parent
S: Jack is a proud father

we arrive at the formal inference

(A) $\mathsf{P},\ ((\mathsf{Q} \land \mathsf{R}) \lor (\neg\mathsf{Q} \land \neg\mathsf{R})),\ (\neg\mathsf{R} \lor \mathsf{S})\ \therefore\ \mathsf{S}$

And of course we can easily find a countervaluation which makes the premisses of this true and conclusion false:

P	Q	R	S	P	$((\mathsf{Q} \land \mathsf{R}) \lor (\neg\mathsf{Q} \land \neg\mathsf{R}))$	$(\neg\mathsf{R} \lor \mathsf{S})$	S
T	F	F	F	T	T	T	F

So (as we would expect) this PL inference isn't tautologically valid (though it is necessarily truth preserving given the meanings of the atoms).

Suppose however, we had adopted a PL language with a different glossary, as follows:

P: Jill is a female
Q: Jill is a parent
R: Jack is a parent
S: Jack is a proud father

And suppose we had done a bit of processing of the claim that Jill is a mother, so we render our argument into this revised language as follows:

(B) $(\mathsf{P} \land \mathsf{Q}),\ ((\mathsf{Q} \land \mathsf{R}) \lor (\neg\mathsf{Q} \land \neg\mathsf{R})),\ (\neg\mathsf{R} \lor \mathsf{S})\ \therefore\ \mathsf{S}$

Then a truth-table quickly reveals this inference to be tautologically valid:

P	Q	R	S	$(\mathsf{P} \land \mathsf{Q})$	$((\mathsf{Q} \land \mathsf{R}) \lor (\neg\mathsf{Q} \land \neg\mathsf{R}))$	$(\neg\mathsf{R} \lor \mathsf{S})$	S
T	T	T	T				T
T	T	T	F	T		F	F
T	T	F	T				T
T	T	F	F	T	F	T	F
T	F	T	T				T
T	F	T	F	F			F
T	F	F	T				T
T	F	F	F	F			F
F	T	T	T				T
F	T	T	F	F			F
F	T	F	T				T
F	T	F	F	F			F
F	F	T	T				T
F	F	T	F	F			F
F	F	F	T				T
F	F	F	F	F			F
				2	4	3	1

First we evaluated the conclusion on every line, and that immediately ruled out eight lines as candidate bad lines where the premisses are true and conclusion false. So next look at the first premiss, and we can immediate rule out another six lines. We are left with just two candidate bad lines with the only examined premiss true and conclusion false. Evaluating the third premiss eliminates another line. So at the fourth and final stage, we only have to look at the middle premiss on one line – and it is false there. So there is no way of making the premisses true and conclusion false. Our second PL inference *is* tautologically valid.

Which is the right way to render our argument into a PL language? As (A), where the rendition is not tautologically valid (thought it is of course valid)? Or as (B) when the rendition *is* tautologically valid?

Well, there *is* no uniquely right answer here. Rather the fact that we can go either way is what's revealing. Taking route (A) we expose that our argument is not valid just in virtue of the meaning of the connectives (as we already knew). But in analysing the idea of Jill's being a mother as a combination of the idea of her being female and a parent, and then taking route (B), we see that that our argument *is* valid on the basis of that analysis plus the meanings of the connectives.

(6) *Either Popper is a logician or Quine is; moreover it isn't the case that either Russell or Sellars are logicians. Why so? Well, for a start, Popper and Russell are logicians. But not both Quine and Sellars are. And finally, this much is ruled out: Sellars not being a logician while Quine is one, and at the same time Russell being a logician too!*

Adopt a PL language with the obvious glossary,

P: Popper is a logician
Q: Quine is a logician
R: Russell is a logician
S: Sellars is a logician

Then we see the core argument comes to this:

$$(\mathsf{P} \land \mathsf{R})\,,\ \neg(\mathsf{Q} \land \mathsf{S}),\ \neg((\neg\mathsf{S} \land \mathsf{Q}) \land \mathsf{R})\ \therefore\ ((\mathsf{P} \lor \mathsf{Q}) \land \neg(\mathsf{R} \lor \mathsf{S})).$$

And if you've been paying attention, you will realize that we showed that this inference (with the premisses in a different order) is tautologically invalid in §15.4(c). If you do not have *IFL2* to hand, here's the truth-table again:

P Q R S	$\neg((\neg\mathsf{S} \land \mathsf{Q}) \land \mathsf{R})$	$(\mathsf{P} \land \mathsf{R})$	$\neg(\mathsf{Q} \land \mathsf{S})$	$((\mathsf{P} \lor \mathsf{Q}) \land \neg(\mathsf{R} \lor \mathsf{S}))$
T T T T		T	F	
T T T F	F T T	T	T	F T
T T F T		F		
T T F F		F		
T F T T	T F F	T	T	F T
T F T F	?	T	T	F T
T F F T		F		
T F F F		F		
F T T T		F		
F T T F		F		
F T F T		F		
F T F F		F		
F F T T		F		
F F T F		F		
F F F T		F		
F F F F		F		
	4	1	2	3

Step (1) already finds twelve lines to be 'good'. At step (2), evaluating the next simplest premiss, another line turns out to be good. Next, we have recorded in miniature that $(\mathsf{R} \lor \mathsf{S})$ is true on each remaining line; so its negation – the second conjunct of the conclusion – is false; so step (3) the whole conclusion is then false. That leaves three lines still in the running as to be bad. So, at step (4), we start work evaluating the first premiss on those three lines; we have again recorded some of our working in miniature as we evaluate that premiss.

On our first potentially bad line, the second line of the table, this premiss comes out false, hence the line is good after all. So we turn to the next potentially bad line, the fifth line of the table. This time the first premiss *is* true: hence we *have* found a bad line where the premisses are all true and conclusion false. And now we can stop, for one bad line is enough.

In sum, then, our argument is not tautologically valid. And it plainly isn't valid for any other reason.