Peter Smith, *Introduction to Formal Logic* (CUP, 2nd edition)

Exercises 20

(a) *Show that the following inferences (in suitable languages, of course) can be warranted by proofs using our rules for conjunction and negation:*

(1) $\mathsf{P},\ \mathsf{Q},\ \mathsf{R}\ \therefore\ (\mathsf{P} \land (\mathsf{P} \land (\mathsf{Q} \land \mathsf{Q})))$

We first prove $(\mathsf{Q} \land \mathsf{Q})$ and then build up to the longer target conclusion by two more applications of conjunction-introduction.

(1)	P	(Prem)
(2)	Q	(Prem)
(3)	R	(Prem)
(4)	$(\mathsf{Q} \land \mathsf{Q})$	($\land$I 2, 2)
(5)	$(\mathsf{P} \land (\mathsf{Q} \land \mathsf{Q}))$	($\land$I 1, 4)
(6)	$(\mathsf{P} \land (\mathsf{P} \land (\mathsf{Q} \land \mathsf{Q})))$	($\land$I 1, 5)

Note that line (4) is legitimate! Our ($\land$I) rule says: Given α and β, we can derive $(\alpha \land \beta)$. Nothing says that α and β have to be distinct: we can derive a conjunction as long as both conjuncts are available. As they are at (4).

(2) $(\mathsf{P} \land (\mathsf{Q} \land \mathsf{R}))\ \therefore\ ((\mathsf{Q} \land \neg\neg\mathsf{P}) \land \mathsf{R})$

Obviously we need to extract all the conjuncts from the premiss, if we are to make use of them. Then we prove $\neg\neg\mathsf{P}$ from P in the hopefully familiar way. Finally we put together the ingredients to get the target conjunction using conjunction-introduction:

(1)	$(\mathsf{P} \land (\mathsf{Q} \land \mathsf{R}))$	(Prem)
(2)	P	($\land$E 1)
(3)	$(\mathsf{Q} \land \mathsf{R})$	($\land$E 1)
(4)	Q	($\land$E 3)
(5)	R	($\land$E 3)
(6)	$\mid\ \neg\mathsf{P}$	(Supp)
(7)	$\mid\ \bot$	(Abs 2, 6)
(8)	$\neg\neg\mathsf{P}$	(RAA 6–7)
(9)	$(\mathsf{Q} \land \neg\neg\mathsf{P})$	($\land$I 4, 8)
(10)	$((\mathsf{Q} \land \neg\neg\mathsf{P}) \land \mathsf{R})$	($\land$I 9, 5)

(3) $(\mathsf{P} \land \mathsf{Q}),\ \neg(\mathsf{P} \land \mathsf{R}),\ \neg(\mathsf{Q} \land \mathsf{S})\ \therefore\ (\neg\mathsf{R} \land \neg\mathsf{S})$

Our target conclusion is a conjunction. The natural way to proceed is to aim to prove each conjunct separately.

So, first, how do we prove $\neg\mathsf{R}$? This new interim target conclusion is a negated wff, and again we do the natural thing – i.e. suppose the opposite and try to derive a contradiction. But that's easy! Then $\neg\mathsf{S}$ is proved similarly, and we arrive at the following proof:

(1)	$(\mathsf{P} \wedge \mathsf{Q})$	(Prem)
(2)	$\neg(\mathsf{P} \wedge \mathsf{R})$	(Prem)
(3)	$\neg(\mathsf{Q} \wedge \mathsf{S})$	(Prem)
(4)	$\quad \mid\ \mathsf{R}$	(Supp)
(5)	$\quad \mid\ \mathsf{P}$	($\wedge$E 1)
(6)	$\quad \mid\ (\mathsf{P} \wedge \mathsf{R})$	($\wedge$1 5, 4)
(7)	$\quad \mid\ \bot$	(Abs 6, 2)
(8)	$\neg\mathsf{R}$	(RAA 5–7)
(9)	$\quad \mid\ \mathsf{S}$	(Supp)
(10)	$\quad \mid\ \mathsf{Q}$	($\wedge$E 1)
(11)	$\quad \mid\ (\mathsf{Q} \wedge \mathsf{S})$	($\wedge$1 10, 9)
(12)	$\quad \mid\ \bot$	(Abs 11, 3)
(13)	$\neg\mathsf{S}$	(RAA 8–12)
(14)	$(\neg\mathsf{R} \wedge \neg\mathsf{S})$	($\wedge$ 8, 13)

Of course, we could have derived P and Q earlier, e.g. at lines 4 and 5!

(4) $\neg(\mathsf{P} \wedge \neg\mathsf{Q}),\ \neg(\mathsf{Q} \wedge \neg\neg\mathsf{R})\ \therefore\ \neg(\mathsf{P} \wedge \neg\neg\mathsf{R})$

The target conclusion is a negated wff. So let's aim for a reductio proof of the following overall shape:

$\neg(\mathsf{P} \wedge \neg\mathsf{Q})$	(Prem)
$\neg(\mathsf{Q} \wedge \neg\neg\mathsf{R})$	(Prem)
$\quad \mid\ (\mathsf{P} \wedge \neg\neg\mathsf{R})$	(Supp)
$\quad \mid\ \vdots$	
$\quad \mid\ \bot$	(Abs)
$\neg(\mathsf{P} \wedge \neg\neg\mathsf{R})$	(RAA)

The supposition gives us P; but we can't use that by itself. If we now also assume $\neg\mathsf{Q}$, however, we can then derive Q in a familiar way using the first premiss (what else?). And then $\bot$ quickly follows using the second premiss:

(1)	$\neg(\mathsf{P} \wedge \neg\mathsf{Q})$	(Prem)
(2)	$\neg(\mathsf{Q} \wedge \neg\neg\mathsf{R})$	(Prem)
(3)	$\quad \mid\ (\mathsf{P} \wedge \neg\neg\mathsf{R})$	(Supp)
(4)	$\quad \mid\ \mathsf{P}$	($\wedge$E 3)
(5)	$\quad \mid\ \quad \mid\ \neg\mathsf{Q}$	(Supp)
(6)	$\quad \mid\ \quad \mid\ (\mathsf{P} \wedge \neg\mathsf{Q})$	($\wedge$I 4, 5)
(7)	$\quad \mid\ \quad \mid\ \bot$	(Abs 6, 1)
(8)	$\quad \mid\ \neg\neg\mathsf{Q}$	(RAA 5–7)
(9)	$\quad \mid\ \mathsf{Q}$	(DN 8)
(10)	$\quad \mid\ \neg\neg\mathsf{R}$	($\wedge$E 3)
(11)	$\quad \mid\ (\mathsf{Q} \wedge \neg\neg\mathsf{R})$	($\wedge$I 9, 10)
(12)	$\quad \mid\ \bot$	(Abs 11, 2)
(13)	$\neg(\mathsf{P} \wedge \neg\neg\mathsf{R})$	(RAA 3–12)

(5) $\neg((\mathsf{P} \vee \mathsf{R}) \wedge \neg\neg(\neg\mathsf{S} \wedge \mathsf{Q})),\ \neg\neg(\neg\mathsf{S} \wedge \mathsf{Q}) \ \therefore\ \neg(\mathsf{P} \vee \mathsf{R})$

Note that, on careful inspection, this argument has the shape

$$\neg(\alpha \wedge \beta), \beta \therefore\ \neg\alpha$$

and so is obviously valid: if it isn't the case that α and β are *both* true, and we are given that the second is true, then the first must be false!

A formal proof by (RAA) is easy, and goes exactly as you would expect. Schematically, any proof of the following shape will be correctly formed:

$$
\begin{array}{lll}
(1) & \neg(\alpha \wedge \beta) & \text{(Prem)} \\
(2) & \beta & \text{(Prem)} \\
(3) & \quad \alpha & \text{(Supp)} \\
(4) & \quad (\alpha \wedge \beta) & (\wedge\text{I } 2,\ 3) \\
(5) & \quad \bot & \text{(Abs } 4,\ 1) \\
(6) & \neg\alpha & \text{(RAA } 3\text{–}12)
\end{array}
$$

In the present case, filling in the details, we have the proof:

$$
\begin{array}{lll}
(1) & \neg((\mathsf{P} \vee \mathsf{R}) \wedge \neg\neg(\neg\mathsf{S} \wedge \mathsf{Q})) & \text{(Prem)} \\
(2) & \neg\neg(\neg\mathsf{S} \wedge \mathsf{Q}) & \text{(Prem)} \\
(3) & \quad (\mathsf{P} \vee \mathsf{R}) & \text{(Supp)} \\
(4) & \quad ((\mathsf{P} \vee \mathsf{R}) \wedge \neg\neg(\neg\mathsf{S} \wedge \mathsf{Q})) & (\wedge\text{I } 2,\ 3) \\
(5) & \quad \bot & \text{(Abs } 4,\ 1) \\
(6) & \neg(\mathsf{P} \vee \mathsf{R}) & \text{(RAA } 3\text{–}12)
\end{array}
$$

Note again that we stated our rules for conjunction and negation to apply to wffs quite generally – and so those wffs can themselves, as here, involve logical apparatus other than conjunction and negation.

(6) $\neg(\mathsf{P} \wedge \mathsf{S}),\ \neg(\neg\mathsf{S} \wedge \mathsf{Q}) \ \therefore\ \neg((\mathsf{P} \wedge \mathsf{R}) \wedge \mathsf{Q})$

Another straightforward proof by reductio beckons, with the overall shape:

$$
\begin{array}{ll}
\neg(\mathsf{P} \wedge \mathsf{S}) & \text{(Prem)} \\
\neg(\neg\mathsf{S} \wedge \mathsf{Q}) & \text{(Prem)} \\
\quad ((\mathsf{P} \wedge \mathsf{R}) \wedge \mathsf{Q}) & \text{(Supp)} \\
\quad \vdots & \\
\quad \bot & \\
\neg((\mathsf{P} \wedge \mathsf{R}) \wedge \mathsf{Q}) & \text{(RAA)}
\end{array}
$$

How are we going to fill in the dots? Obviously we need to disassemble the conjunction at the third line. Evidently R will do no work; but P should combine with the first premiss and Q with the second premiss to give contradictory results. So here's one way of spelling that out:

(1)	$\neg(\mathsf{P} \wedge \mathsf{S})$	(Prem)
(2)	$\neg(\neg\mathsf{S} \wedge \mathsf{Q})$	(Prem)
(3)	$\quad ((\mathsf{P} \wedge \mathsf{R}) \wedge \mathsf{Q})$	(Supp)
(4)	$\quad (\mathsf{P} \wedge \mathsf{R})$	($\wedge$E 3)
(5)	$\quad \mathsf{Q}$	($\wedge$E 3)
(6)	$\quad \mathsf{P}$	($\wedge$E 4)
(7)	$\quad\quad \mathsf{S}$	(Supp)
(8)	$\quad\quad (\mathsf{P} \wedge \mathsf{S})$	($\wedge$I 6, 7)
(9)	$\quad\quad \bot$	(Abs 8, 1)
(10)	$\quad \neg\mathsf{S}$	(RAA 7–9)
(11)	$\quad (\neg\mathsf{S} \wedge \mathsf{Q})$	($\wedge$I 10, 5)
(12)	$\quad \bot$	(Abs 11, 2)
(13)	$\neg((\mathsf{P} \wedge \mathsf{R}) \wedge \mathsf{Q})$	(RAA 3–12)

(7) $\neg(\mathsf{P} \wedge \neg(\mathsf{S} \wedge \mathsf{Q}))$, $(\neg\mathsf{R} \wedge \neg\neg\mathsf{P})$ $\therefore$ $(\mathsf{Q} \wedge \neg\neg\neg\mathsf{R})$

This is as much as anything a little exercise in recognizing patterns of inference which you've seen before. You know how to prove that α entails $\neg\neg\alpha$; so of course, you can show $\neg\mathsf{R}$ entails $\neg\neg\neg\mathsf{R}$. You know how to prove that α and $\neg(\alpha \wedge \neg\beta)$ entails β. So you can show that P (or equivalently $\neg\neg\mathsf{P}$)) and $\neg(\mathsf{P} \wedge \neg(\mathsf{S} \wedge \mathsf{Q}))$ entail $(\mathsf{S} \wedge \mathsf{Q})$ and hence Q. We just need to put things together!

(1)	$\neg(\mathsf{P} \wedge \neg(\mathsf{S} \wedge \mathsf{Q}))$	(Prem)
(2)	$(\neg\mathsf{R} \wedge \neg\neg\mathsf{P})$	(Prem)
(3)	$\neg\mathsf{R}$	($\wedge$E 2)
(4)	$\quad \neg\neg\mathsf{R}$	(Supp)
(5)	$\quad \bot$	(Abs 3, 4)
(6)	$\neg\neg\neg\mathsf{R}$	(RAA 3–5)
(7)	$\neg\neg\mathsf{P}$	($\wedge$E 2)
(8)	P	(DN 7)
(9)	$\quad \neg(\mathsf{S} \wedge \mathsf{Q})$	(Supp)
(10)	$\quad (\mathsf{P} \wedge \neg(\mathsf{S} \wedge \mathsf{Q}))$	($\wedge$I 8, 9)
(11)	$\quad \bot$	(Abs 10, 1)
(12)	$\neg\neg(\mathsf{S} \wedge \mathsf{Q})$	(RAA 9–11)
(13)	$(\mathsf{S} \wedge \mathsf{Q})$	(DN 12)
(14)	Q	($\wedge$E 13)
(15)	$(\mathsf{Q} \wedge \neg\neg\neg\mathsf{R})$	($\wedge$I 14, 6)

(8) $\neg(\mathsf{P} \wedge \mathsf{S})$, $\neg(\neg\mathsf{S} \wedge \mathsf{Q})$, $((\mathsf{P} \wedge \mathsf{R}) \wedge \mathsf{Q})$ $\therefore$ P' How can this be valid? How can the atom P', unrelated to any of the premisses, follow from them? It must be because they are inconsistent, prove $\bot$, and then we can derive anything, including P', by (EFQ).

So how do we prove the premisses are inconsistent? You've in effect just done that in answering Qn 6! There you showed that the first two premisses plus the third wff treated as a supposition lead to contradiction. The same line of proof will now work again, with the previously indented column shifted over (because we are dealing now with a third premiss, not a supposition). So here we go!

(1)	$\neg(\mathsf{P} \wedge \mathsf{S})$	(Prem)
(2)	$\neg(\neg\mathsf{S} \wedge \mathsf{Q})$	(Prem)
(3)	$((\mathsf{P} \wedge \mathsf{R}) \wedge \mathsf{Q})$	(Prem)
(4)	$(\mathsf{P} \wedge \mathsf{R})$	($\wedge$E 3)
(5)	Q	($\wedge$E 3)
(6)	P	($\wedge$E 4)
(7)	S	(Supp)
(8)	$(\mathsf{P} \wedge \mathsf{S})$	($\wedge$I 6, 7)
(9)	$\bot$	(Abs 8, 1)
(10)	$\neg\mathsf{S}$	(RAA 7–9)
(11)	$(\neg\mathsf{S} \wedge \mathsf{Q})$	($\wedge$I 10, 5)
(12)	$\bot$	(Abs 11, 2)
(13)	P'	(EFQ 12)

(9) $\neg(\mathsf{P} \wedge \neg\neg\neg\bot) \;\therefore\; \neg\mathsf{P}$

This is straightforward. We obviously need another reductio proof. So we will suppose P. To get a contradiction with the first premiss, we need to prove $\neg\neg\neg\bot$. Do it by reductio! Assume $\neg\neg\bot$, and then straightaway we can derive a contradiction.

(1)	$\neg(\mathsf{P} \wedge \neg\neg\neg\bot)$	(Prem)
(2)	P	(Supp)
(3)	$\neg\neg\bot$	(Supp)
(4)	$\bot$	(DN 3)
(5)	$\neg\neg\neg\bot$	(RAA 3–4)
(6)	$(\mathsf{P} \wedge \neg\neg\neg\bot)$	($\wedge$I 2, 5)
(7)	$\bot$	(Abs6, 1)
(8)	$\neg\mathsf{P}$	(RAA 2–7)

(10) $(\mathsf{P} \to \mathsf{Q}) \;\therefore ((\mathsf{P} \to \mathsf{Q}) \;\wedge \neg\bot)$

Don't be distracted by the conditional here: it is doing no work! Any inference of the form $\alpha \;\therefore (\alpha \;\wedge \neg\bot)$ is valid. Why? Because $\neg\bot$ is a logical truth and we should always be able to derive it when we want to. But how can we do that? Well, consider this proof-fragment:

$\bot$	(Supp)
$\vdots$	
$\bot$	
$\neg\bot$	(RAA)

How do we fill in the dots? In the next chapter, we will add an 'iteration' rule which allows us to repeat (accessible) wffs; but we haven't got that yet. To use the existing rules to take us from $\bot$ 'above the line' (as a supposition) to $\bot$ 'below the line' (as a conclusion of the subproof) so that we can use our version of (RAA) we'll need to use a bit of trickery, like this for example:

$$\begin{array}{l|l}
\quad\quad \underline{\bot} & \text{(Supp)} \\
\quad\quad (\bot \land \alpha) & (\land\text{I}) \\
\quad\quad \bot & (\land\text{E}) \\
\neg\bot & \text{(RAA)}
\end{array}$$

Here α can be any other available wff or can indeed be $\bot$ again.

So here's a proof for (10):

$$\begin{array}{lll}
(1) & (\mathsf{P} \to \mathsf{Q}) & \text{(Prem)} \\
(2) & \quad\quad \underline{\bot} & \text{(Supp)} \\
(3) & \quad\quad (\bot \land (\mathsf{P} \to \mathsf{Q})) & (\land\text{I 2, 1}) \\
(4) & \quad\quad \bot & (\land\text{E 3}) \\
(5) & \neg\bot & \text{(RAA 2–4)} \\
(6) & ((\mathsf{P} \to \mathsf{Q}) \ \land \neg\bot) & (\land\text{I 1, 5})
\end{array}$$

So far we have been producing particular formal proofs *inside* our Fitch-style proof system for handling arguments which turn on negation and conjunction. The next set of questions – the first of a series of exercises in the next few chapters – asks us to (as it were) go up a level and informally establish general results *about* our proof system. (In jargon we'll meet properly later, we are going we are doing a little light 'metatheory'.)

Do not worry at all if you (initially) find this more abstract sort of question a bit harder, or are not (initially) confident about what counts as a satisfying answer. In fact it is only proofs in the (optional!) Appendix that will directly depend on the little results we mention here. But for all that, it is worth trying to follow the answers, to get a better sense of how our Fitch-style system works.

(b*) *Recall the 'Γ' notation from Exercises 16(c*), introduced to indicate some wffs (zero, one, or many), with 'Γ, α' indicating those wffs together with α. And recall the use of 'iff' introduced in §18.6. We now add a new pair of definitions*

> Γ *are S*-consistent *– i.e., are consistent as far as the proof system S can tell – iff there is no proof in system S of $\bot$ from Γ as premisses.*
>
> Γ *are S*-inconsistent *iff there is an S-proof of $\bot$ from Γ as premisses.*

Let S be the current proof system with our conjunction and negation rules. Show:

(1) α can be derived in S from Γ as premisses iff $\Gamma, \neg\alpha$ are S-inconsistent.

Just reflect that if there is a proof of the shape on the left below, then if we add $\neg\alpha$ as an additional premiss at the start, we can repeat the proof as before to derive α, and then add another line to get absurdity, as on the right:

$$\begin{array}{|ll}
\Gamma & \text{(a list of premisses)} \\
\hline
\vdots & \\
\alpha &
\end{array}
\qquad\qquad
\begin{array}{|ll}
\Gamma & \text{(a list of premisses)} \\
\neg\alpha & \text{(another premiss)} \\
\hline
\vdots & \\
\alpha & \\
\bot & \text{(using the extra premiss)}
\end{array}$$

So indeed, if (as on the left) α can be derived in S from Γ, then (as shown on the right) $\Gamma, \neg\alpha$ are S-inconsistent.

For the converse, note that given a proof as on the left below, we could indent the lines from $\neg\alpha$ on, and then get the proof on the right:

$$\begin{array}{|ll}
\Gamma & \text{(list of premisses)} \\
\neg\alpha & \text{(another premiss)} \\
\hline
\vdots & \\
\bot &
\end{array}
\qquad\qquad
\begin{array}{|ll}
\Gamma & \text{(list of premisses)} \\
\hline
\quad\begin{array}{|l}\neg\alpha \\ \hline \vdots \\ \bot\end{array} & \text{(now a supposition)} \\
\neg\neg\alpha & \text{(RAA)} \\
\alpha & \text{(DN)}
\end{array}$$

So indeed, if $\Gamma, \neg\alpha$ are S-inconsistent (i.e. prove $\bot$, as on the left) then α can be derived from Γ (as on the right).

Now for three results (for eventual use in the Appendix) about what we can add *to S-consistent wffs while keeping them S-consistent. First, note that if Γ, α are S-inconsistent, Γ proves $\neg\alpha$; so if Γ, α are S-inconsistent and $\neg\neg\alpha$ is one of the wffs Γ, then Γ must already be S-inconsistent. (Explain why!)*

That's elementary! Using the same sort of argument as for (1), if Γ, α are S-inconsistent, there is a proof from Γ to $\neg\alpha$. If $\neg\neg\alpha$ is already one of the wffs Γ, we can just add one more step to that proof (applying (Abs) to the available wffs $\neg\alpha$ and $\neg\neg\alpha$) to get a proof deriving $\bot$ just from Γ.

Conclude that

(2) If the wffs Γ are S-consistent and $\neg\neg\alpha$ is one of them, then Γ, α are also S-consistent.

This is just straightforward propositional reasoning from what we have just shown. If that isn't clear, put:

P: The wffs Γ are S-consistent.
Q: $\neg\neg\alpha$ is one of the wffs Γ.
R: The wffs Γ, α are S-consistent.

Then the elementary point we have already just shown is that

(*) If *not-R* and Q, then *not-P*

and what we now want to conclude is that

(2) If P and Q, then R.

And that indeed follows – either do a truth-table(!), or note that if we had P and Q and also *not-R*, we would have a contradiction using (*).

We use 'Γ, α, β' to indicate the wffs Γ together with α and β. Show that

(3) If the wffs Γ are S-consistent and $(\alpha \wedge \beta)$ is one of them, then Γ, α, β are also S-consistent.

Suppose Γ, α, β are S-inconsistent. Then if Γ contains $(\alpha \wedge \beta)$, we have a proof from Γ on to α and to β, and by our supposition we can now extend that proof to a proof of $\perp$.

In other words, if Γ, α, β are S-inconsistent, and $(\alpha \wedge \beta)$ is one of Γ, then Γ is S-inconsistent. Re-arranging that as in the argument for (2) gives us (3).

Note too that if $\Gamma, \neg\alpha$ and $\Gamma, \neg\beta$ are both S-inconsistent, we can derive both α and β from Γ, and hence can derive $(\alpha \wedge \beta)$. So if $\Gamma, \neg\alpha$ and $\Gamma, \neg\beta$ are both S-inconsistent and these wffs Γ already include $\neg(\alpha \wedge \beta)$, then Γ are S-inconsistent (why?).

This is elementary. If $\Gamma, \neg\alpha$ and $\Gamma, \neg\beta$ are both S-inconsistent, then – by the first point – we can derive $(\alpha \wedge \beta)$ from Γ and so if Γ already include $\neg(\alpha \wedge \beta)$ we could appeal to that premiss and go on another step to derive $\perp$.

Conclude

(4) If the wffs Γ are S-consistent and $\neg(\alpha \wedge \beta)$ is one of them, then either $\Gamma, \neg\alpha$ or $\Gamma, \neg\beta$ (or both) are also S-consistent.

This is just propositional reasoning from the point just made. If you don't see that immediately, put

P: The wffs Γ are S-consistent.
Q: $\neg(\alpha \wedge \beta)$ is one of the wffs Γ.
R: The wffs $\Gamma, \neg\alpha$ are S-consistent.
S: The wffs $\Gamma, \neg\beta$ are S-consistent.

Then the elementary point we have already just shown is that

(**) If *not-R* and *not-S* and Q, then *not-P*

and what we want to conclude is that

(4) If P and Q, then either R or S.

And that indeed follows – either do a truth-table(!), or note that if we had P and Q and neither R nor S, we would have a contradiction using (**).