Peter Smith, *Introduction to Formal Logic* (CUP, 2nd edition)

Exercises 21

(a) *Show that the following inferences can be warranted by proofs using the derivation rules introduced over the last two chapters:*

Let's again be clear before we start. Understanding how proofs work is the crucial task, at least for philosophy students learning some logic. And getting familiar with some recurrent patterns of arguments is of course helpful in promoting such understanding. But becoming really good at *proof-discovery* should really be thought of more as an optional extra than as an essential skill that every beginning logic student needs to master.

Still, in giving the answers below, we'll try to explain the strategic thinking – and hopefully, at least in these simple cases, you will see that the required proofs often come near to writing themselves.

(1) $(\mathsf{P} \lor (\mathsf{Q} \land \mathsf{R})) \;\therefore\; ((\mathsf{P} \lor \mathsf{Q}) \land (\mathsf{P} \lor \mathsf{R}))$

The conclusion is a conjunction: so the obvious strategy is to try to prove each conjunct separately. But, for example, $(\mathsf{P} \lor \mathsf{Q})$ follows from $(\mathsf{P} \lor (\mathsf{Q} \land \mathsf{R}))$ just arguing by cases. From P we trivially get $(\mathsf{P} \lor \mathsf{Q})$. Equally, from $(\mathsf{Q} \land \mathsf{R})$ we get Q and hence $(\mathsf{P} \lor \mathsf{Q})$. So either way, we get the desired first conjunct of our conclusion. The second conjunct follows similarly. So we just need to put everything together!

(1)	$(\mathsf{P} \lor (\mathsf{Q} \land \mathsf{R}))$	(Prem)
(2)	$\quad\mathsf{P}$	(Supp)
(3)	$\quad(\mathsf{P} \lor \mathsf{Q})$	($\lor$I 2)
(4)	$\quad(\mathsf{Q} \land \mathsf{R})$	(Supp)
(5)	$\quad\mathsf{Q}$	($\land$E 4)
(6)	$\quad(\mathsf{P} \lor \mathsf{Q})$	($\lor$I 5)
(7)	$(\mathsf{P} \lor \mathsf{Q})$	($\lor$E 1, 2–3, 4–6)
(8)	$\quad\mathsf{P}$	(Supp)
(9)	$\quad(\mathsf{P} \lor \mathsf{R})$	($\lor$I 8)
(10)	$\quad(\mathsf{Q} \land \mathsf{R})$	(Supp)
(11)	$\quad\mathsf{R}$	($\land$E 10)
(12)	$\quad(\mathsf{P} \lor \mathsf{R})$	($\lor$I 11)
(13)	$(\mathsf{P} \lor \mathsf{R})$	($\lor$E 1, 8–9, 10–12)
(14)	$((\mathsf{P} \lor \mathsf{Q}) \land (\mathsf{P} \lor \mathsf{R}))$	($\land$I 7, 13)

But that proof-strategy is a bit uneconomical – look how we made the same supposition at (2) and (8), and again made the same supposition at (4) at (10). There's nothing *wrong* in doing that; the proof we have just given is perfectly in accordance with our proof-building rules of inference. Still, can we give a more economical argument? Well, yes:

(1)	$\mid\ \underline{(P \lor (Q \land R))}$	(Prem)
(2)	$\mid\ \mid\ \underline{P}$	(Supp)
(3)	$\mid\ \mid\ (P \lor Q)$	($\lor$I 2)
(4)	$\mid\ \mid\ (P \lor R)$	($\lor$I 2)
(5)	$\mid\ \mid\ ((P \lor Q) \land (P \lor R))$	($\land$I 3, 4)
(6)	$\mid\ \mid\ \underline{(Q \land R)}$	(Supp)
(7)	$\mid\ \mid\ Q$	($\land$E 6)
(8)	$\mid\ \mid\ R$	($\land$E 6)
(9)	$\mid\ \mid\ (P \lor Q)$	($\lor$I 7)
(10)	$\mid\ \mid\ (P \lor R)$	($\lor$I 8)
(11)	$\mid\ \mid\ ((P \lor Q) \land (P \lor R))$	($\land$I 9, 10)
(12)	$\mid\ ((P \lor Q) \land (P \lor R))$	($\lor$E 1, 2–5, 6–11)

(2) $(P \lor Q) \ \therefore\ \neg(\neg P \land \neg Q)$

This is an instance of one direction of one of De Morgan's Laws. Since the target is the negation of something, let's assume the opposite and aim for a contradiction. So we will expect the overall shape of the proof to be:

$\mid\ \underline{(P \lor Q)}$	(Prem)
$\mid\ \mid\ \underline{(\neg P \land \neg Q)}$	(Supp)
$\mid\ \mid\ \neg P$	($\land$E)
$\mid\ \mid\ \neg Q$	($\land$E)
$\mid\ \mid\ \vdots$	
$\mid\ \mid\ \bot$	
$\mid\ \neg(\neg P \land \neg Q)$	(RAA)

To fill in the rest of the proof, we obviously need to use our only premiss in an argument-by-cases, using ($\lor$E). So we will expect the shape of the argument to be filled out like this:

$\mid\ \underline{(P \lor Q)}$	(Prem)
$\mid\ \mid\ \underline{(\neg P \land \neg Q)}$	(Supp)
$\mid\ \mid\ \neg P$	($\land$E)
$\mid\ \mid\ \neg Q$	($\land$E)
$\mid\ \mid\ \mid\ \underline{P}$	(Supp)
$\mid\ \mid\ \mid\ \vdots$	
$\mid\ \mid\ \mid\ \bot$	
$\mid\ \mid\ \mid\ \underline{Q}$	(Supp)
$\mid\ \mid\ \mid\ \vdots$	
$\mid\ \mid\ \mid\ \bot$	
$\mid\ \mid\ \bot$	($\lor$E)
$\mid\ \neg(\neg P \land \neg Q)$	(RAA)

But hold on! We see that there is zero work to be done 'filling in the dots' as in each subproof the absurdity already follows from available wffs! So we are in fact already done:

(1)	$(\mathsf{P} \lor \mathsf{Q})$	(Prem)
(2)	$\quad (\neg\mathsf{P} \land \neg\mathsf{Q})$	(Supp)
(3)	$\quad \neg\mathsf{P}$	($\land$E 2)
(4)	$\quad \neg\mathsf{Q}$	($\land$E 2)
(5)	$\quad\quad \mathsf{P}$	(Supp)
(6)	$\quad\quad \bot$	(Abs 5, 3)
(7)	$\quad\quad \mathsf{Q}$	(Supp)
(8)	$\quad\quad \bot$	(Abs 7, 4)
(9)	$\quad \bot$	($\lor$E 1 5–6, 7–8)
(10)	$\neg(\neg\mathsf{P} \land \neg\mathsf{Q})$	(RAA 2–9)

(3) $\neg(\mathsf{P} \land \mathsf{Q}) \ \therefore\ (\neg\mathsf{P} \lor \neg\mathsf{Q})$

Short answer: you know from $\mathbf{E}'$ in the body of the chapter how to argue from $\neg(\neg\mathsf{P} \lor \neg\mathsf{Q})$ to $(\mathsf{P} \land \mathsf{Q})$. And in general, if you have a proof [A] from $\neg\alpha$ to β, you can construct a proof [B] from $\neg\beta$ to α like this!

$\neg\beta$	(Prem)
$\quad \neg\alpha$	(Supp)
$\quad \vdots$	
$\quad \beta$	(from the supposition by the assumed proof [A])
$\quad \bot$	(from β and $\neg\beta$)
$\neg\neg\alpha$	(RAA)
$\bot$	(DN)

But let's think through the needed proof again from scratch.

We have no useful rule we can directly apply to the premiss (what would be the point of reiterating it or conjoining it with itself?). And evidently there can be no way of getting from the premiss to one of the disjuncts in the conclusion. Our only hope, then is use a reductio argument – i.e. we assume the opposite of the conclusion and aim to derive a contradiction:

$\neg(\mathsf{P} \land \mathsf{Q})$	(Prem)
$\quad \neg(\neg\mathsf{P} \lor \neg\mathsf{Q})$	(Supp)
$\quad \vdots$	
$\quad \bot$	
$\neg\neg(\neg\mathsf{P} \lor \neg\mathsf{Q})$	(RAA)
$(\neg\mathsf{P} \lor \neg\mathsf{Q})$	(DN)

But again we *still* have no useful rule we can directly apply to the original premiss and/or the new supposition (what would be the point of conjoining them, for example?). So our only hope is to make *another*, new, supposition. But what?

We need something which will somehow link up with the wffs already in the proof. It is often a good strategy in such a case to try a component of a wff we already have. Supposing P, however, fairly obviously won't be much use: so let's try $\neg\mathsf{P}$. Then we can make progress like this:

$$
\begin{array}{ll}
\mid \underline{\neg(\mathsf{P} \land \mathsf{Q})} & \text{(Prem)} \\
\mid \quad \mid \underline{\neg(\neg\mathsf{P} \lor \neg\mathsf{Q})} & \text{(Supp)} \\
\mid \quad \mid \quad \mid \underline{\neg\mathsf{P}} & \text{(Supp)} \\
\mid \quad \mid \quad \mid (\neg\mathsf{P} \lor \neg\mathsf{Q}) & (\lor\text{I}) \\
\mid \quad \mid \quad \mid \bot & \text{(Abs)} \\
\mid \quad \mid \neg\neg\mathsf{P} & \text{(RAA)} \\
\mid \quad \mid \mathsf{P} & \text{(DN)} \\
\mid \quad \mid \vdots &
\end{array}
$$

And now we can see how to complete the proof. For we can similarly prove Q, and hence get $(\mathsf{P} \land \mathsf{Q})$ – which yields our desired contradiction with the initial premiss.

So finishing the proof to that plan we get:

$$
\begin{array}{lll}
(1) & \mid \underline{\neg(\mathsf{P} \land \mathsf{Q})} & \text{(Prem)} \\
(2) & \mid \quad \mid \underline{\neg(\neg\mathsf{P} \lor \neg\mathsf{Q})} & \text{(Supp)} \\
(3) & \mid \quad \mid \quad \mid \underline{\neg\mathsf{P}} & \text{(Supp)} \\
(4) & \mid \quad \mid \quad \mid (\neg\mathsf{P} \lor \neg\mathsf{Q}) & (\lor\text{I } 3) \\
(5) & \mid \quad \mid \quad \mid \bot & \text{(Abs 4, 2)} \\
(6) & \mid \quad \mid \neg\neg\mathsf{P} & \text{(RAA 3–5)} \\
(7) & \mid \quad \mid \mathsf{P} & \text{(DN 6)} \\
(8) & \mid \quad \mid \quad \mid \underline{\neg\mathsf{Q}} & \text{(Supp)} \\
(9) & \mid \quad \mid \quad \mid (\neg\mathsf{P} \lor \neg\mathsf{Q}) & (\lor\text{I } 8) \\
(10) & \mid \quad \mid \quad \mid \bot & \text{(Abs 9, 2)} \\
(11) & \mid \quad \mid \neg\neg\mathsf{Q} & \text{(RAA 8–10)} \\
(12) & \mid \quad \mid \mathsf{Q} & \text{(DN 11)} \\
(13) & \mid \quad \mid (\mathsf{P} \land \mathsf{Q}) & (\land\text{I } 7, 12) \\
(14) & \mid \quad \mid \bot & \text{(Abs 13, 1)} \\
(15) & \mid \neg\neg(\neg\mathsf{P} \lor \neg\mathsf{Q}) & \text{(RAA 2–14)} \\
(16) & \mid (\neg\mathsf{P} \lor \neg\mathsf{Q}) & \text{(DN 15)}
\end{array}
$$

(4) $(\mathsf{P} \lor \mathsf{Q}),\ (\mathsf{P} \lor \mathsf{R}) \ \therefore\ (\mathsf{P} \lor (\mathsf{Q} \land \mathsf{R}))$

Obviously we will have to use our two premisses in $(\lor\text{E})$ inferences (and it evidently can't matter which we use first). So, we start off knowing that our proof can have this shape:

$$
\begin{array}{l|lll}
 & (\mathsf{P} \vee \mathsf{Q}) & & \text{(Prem)} \\
 & \underline{(\mathsf{P} \vee \mathsf{R})} & & \text{(Prem)} \\
 & \quad \vert \ \underline{\mathsf{P}} & & \text{(Supp)} \\
 & \quad \vert \ \vdots & & \\
 & \quad \vert \ (\mathsf{P} \vee (\mathsf{Q} \wedge \mathsf{R})) & & \\
 & \quad \vert \ \underline{\mathsf{Q}} & & \text{(Supp)} \\
 & \quad \vert \ \vdots & & \\
 & \quad \vert \ (\mathsf{P} \vee (\mathsf{Q} \wedge \mathsf{R})) & & \\
 & (\mathsf{P} \vee (\mathsf{Q} \wedge \mathsf{R})) & & (\vee\mathrm{E})
\end{array}
$$

But of course, the dots in the first subproof can be eliminated, as we can immediately infer the subproof's conclusion from its supposition using ($\vee$I). What about the second subproof? Here, to fill things out, we will evidently need to appeal to the original second premiss and use *that* in another proof by ($\vee$E), to give us now a proof of the following shape:

$$
\begin{array}{l|lll}
 & (\mathsf{P} \vee \mathsf{Q}) & & \text{(Prem)} \\
 & \underline{(\mathsf{P} \vee \mathsf{R})} & & \text{(Prem)} \\
 & \quad \vert \ \underline{\mathsf{P}} & & \text{(Supp)} \\
 & \quad \vert \ (\mathsf{P} \vee (\mathsf{Q} \wedge \mathsf{R})) & & (\vee\mathrm{I}) \\
 & \quad \vert \ \underline{\mathsf{Q}} & & \text{(Supp)} \\
 & \quad \vert \quad \vert \ \underline{\mathsf{P}} & & \text{(Supp)} \\
 & \quad \vert \quad \vert \ \vdots & & \\
 & \quad \vert \quad \vert \ (\mathsf{P} \vee (\mathsf{Q} \wedge \mathsf{R})) & & \\
 & \quad \vert \quad \vert \ \underline{\mathsf{R}} & & \text{(Supp)} \\
 & \quad \vert \quad \vert \ \vdots & & \\
 & \quad \vert \quad \vert \ (\mathsf{P} \vee (\mathsf{Q} \wedge \mathsf{R})) & & \\
 & \quad \vert \ (\mathsf{P} \vee (\mathsf{Q} \wedge \mathsf{R})) & & \\
 & (\mathsf{P} \vee (\mathsf{Q} \wedge \mathsf{R})) & & (\vee\mathrm{E})
\end{array}
$$

And now it is obvious how to complete the proof:

> The dots in the first incomplete subproof can just be erased, as again its conclusion follows immediately from the subproof's supposition.
> The dots in the second incomplete subproof evidently need to be filled by $(\mathsf{Q} \wedge \mathsf{R})$.

Which gives us the following completed proof, with all the annotations in place. (Do make sure you understand the structure of this proof, and the way that the two applications of ($\vee$I) are nested one inside the other.)

(1)	$\mid (\mathsf{P} \lor \mathsf{Q})$	(Prem)
(2)	$\mid \underline{(\mathsf{P} \lor \mathsf{R})}$	(Prem)
(3)	$\mid \quad \mid \underline{\mathsf{P}}$	(Supp)
(4)	$\mid \quad \mid (\mathsf{P} \lor (\mathsf{Q} \land \mathsf{R}))$	($\lor$I 3)
(5)	$\mid \quad \mid \underline{\mathsf{Q}}$	(Supp)
(6)	$\mid \quad \mid \quad \mid \underline{\mathsf{P}}$	(Supp)
(7)	$\mid \quad \mid \quad \mid (\mathsf{P} \lor (\mathsf{Q} \land \mathsf{R}))$	($\lor$I 6)
(8)	$\mid \quad \mid \quad \mid \underline{\mathsf{R}}$	(Supp)
(9)	$\mid \quad \mid \quad \mid (\mathsf{Q} \land \mathsf{R})$	($\land$E 5, 8)
(10)	$\mid \quad \mid \quad \mid (\mathsf{P} \lor (\mathsf{Q} \land \mathsf{R}))$	($\lor$I 9)
(11)	$\mid \quad \mid (\mathsf{P} \lor (\mathsf{Q} \land \mathsf{R}))$	($\lor$E 2, 6–7, 8–10)
(12)	$\mid (\mathsf{P} \lor (\mathsf{Q} \land \mathsf{R}))$	($\lor$E 1, 3–4, 5–11)

(5) $(\mathsf{P} \lor \bot),\ (\mathsf{Q} \lor \bot)\ \therefore\ (\mathsf{P} \land \mathsf{Q})$

Since $\bot$ is always false, $(\alpha \lor \bot)$ is true when and only when α is true. So we should expect there to be a proof in our Fitch-style system from $(\alpha \lor \bot)$ to α, whatever α is. And there is. We have a disjunctive premiss, so we just need to set off on a simple argument by ($\lor$E):

$\mid \underline{(\alpha \lor \bot)}$	(Prem)
$\mid \quad \mid \underline{\alpha}$	(Supp)
$\mid \quad \mid \alpha$	(Iter)
$\mid \quad \mid \underline{\bot}$	(Supp)
$\mid \quad \mid \alpha$	(EFQ)
$\mid \alpha$	($\lor$E)

So, for (5), we just need to chain together two proofs like this to give

(1)	$\mid (\mathsf{P} \lor \bot)$	(Prem)
(2)	$\mid \underline{(\mathsf{Q} \lor \bot)}$	(Prem)
(3)	$\mid \quad \mid \underline{\mathsf{P}}$	(Supp)
(4)	$\mid \quad \mid \mathsf{P}$	(Iter 3)
(5)	$\mid \quad \mid \underline{\bot}$	(Supp)
(6)	$\mid \quad \mid \mathsf{P}$	(EFQ 5)
(7)	$\mid \mathsf{P}$	($\lor$E 1, 3–4, 5–6)
(8)	$\mid \quad \mid \underline{\mathsf{Q}}$	(Supp)
(9)	$\mid \quad \mid \mathsf{Q}$	(Iter 8)
(10)	$\mid \quad \mid \underline{\bot}$	(Supp)
(11)	$\mid \quad \mid \mathsf{Q}$	(EFQ 10)
(12)	$\mid \mathsf{Q}$	($\lor$E 2, 8–9, 10–11)
(13)	$\mid (\mathsf{P} \land \mathsf{Q})$	($\land$I 7, 12)

(6) $\neg(\mathsf{Q} \land \mathsf{P})$, $((\mathsf{R} \land \mathsf{Q}) \lor (\mathsf{P} \land \mathsf{Q})) \therefore \mathsf{R}$

Suppose the second premiss had instead been $((\mathsf{R} \land \mathsf{Q}) \lor (\mathsf{Q} \land \mathsf{P}))$. Then the premisses would have had the form $\neg\beta, (\alpha \lor \beta)$, and a simple argument by disjunctive syllogism would then give us α, i.e. $(\mathsf{R} \land \mathsf{Q})$, and hence R.

As things stand, however, we have to complicate things just a little; we need to swap the order of P and Q so things match up for the disjunctive syllogism. But that's easy enough!

(1)	$\neg(\mathsf{Q} \land \mathsf{P})$		(Prem)
(2)	$((\mathsf{R} \land \mathsf{Q}) \lor (\mathsf{P} \land \mathsf{Q}))$		(Prem)
(3)		$(\mathsf{R} \land \mathsf{Q})$	(Supp)
(4)		R	($\land$E 3)
(5)		$(\mathsf{P} \land \mathsf{Q})$	(Supp)
(6)		P	($\land$E 5)
(7)		Q	($\land$E 5)
(8)		$(\mathsf{Q} \land \mathsf{P})$	($\land$I 7, 6)
(9)		$\perp$	(Abs 8, 1)
(10)		R	(EFQ 9)
(11)	R		($\lor$E 2, 3–4, 5–10)

(7) $(\mathsf{P} \lor \neg\mathsf{Q})$, $(\mathsf{R} \lor \neg\mathsf{P})$, $(\neg\neg\mathsf{R} \lor \mathsf{Q}) \therefore \mathsf{R}$.

Let's try arguing by cases, using the first premiss and planning to apply ($\lor$E), so needing two subproofs, one from each disjunct of that first premiss.

Then a moment's reflection suggests that we should be able to construct a proof of the following shape:

$(\mathsf{P} \lor \neg\mathsf{Q})$		(Prem)
$(\mathsf{R} \lor \neg\mathsf{P})$		(Prem)
$(\neg\neg\mathsf{R} \lor \mathsf{Q})$		(Prem)
	P	(Supp)
	$\vdots$	(use disj. syll. with the second premiss)
	R	
	$\neg\mathsf{Q}$	(Supp)
	$\vdots$	(use disj. syll. and DN with the third premiss)
	R	
R		($\lor$E from the first premiss and the two subproofs)

Of course, we could equally well argue by cases from the second premiss, giving us a proof which is now structured like this:

$(P \lor \neg Q)$	(Prem)
$(R \lor \neg P)$	(Prem)
$(\neg\neg R \lor Q)$	(Prem)
$\quad R$	(Supp)
$\quad R$	(Iter)
$\quad \neg P$	(Supp)
$\quad \vdots$	(use disj. syll. with the *first* premiss)
$\quad \neg Q$	
$\quad \vdots$	(now use disj. syll. and DN with the *third* premiss)
$\quad R$	(DN)
R	($\lor$E from the *second* premiss and the two subproofs)

And we could equally well argue by cases from the *third* premiss, using disjunctive syllogisms with the first and second premiss! It doesn't matter which way you go.

But for the record, filling in the details of the proof following our first strategy, we would have:

(1)	$(P \lor \neg Q)$	(Prem)
(2)	$(R \lor \neg P)$	(Prem)
(3)	$(\neg\neg R \lor Q)$	(Prem)
(4)	$\quad P$	(Supp)
(5)	$\quad\quad R$	(Supp)
(6)	$\quad\quad R$	(Iter 5)
(7)	$\quad\quad \neg P$	(Supp)
(8)	$\quad\quad \bot$	(Abs 4, 7)
(9)	$\quad\quad R$	(EFQ 8)
(10)	$\quad R$	($\lor$E 2, 5–6, 7–9)
(11)	$\quad \neg Q$	(Supp)
(12)	$\quad\quad \neg\neg R$	(Supp)
(13)	$\quad\quad R$	(DN 12)
(14)	$\quad\quad Q$	(Supp)
(15)	$\quad\quad \bot$	(Abs 14, 11)
(16)	$\quad\quad R$	(EFQ 15)
(17)	$\quad R$	($\lor$E 3, 12–13, 14–16)
(18)	R	($\lor$E 1, 4–10, 11–17)

That's a bit long: but in fact, if you *first* think about the overall strategy and only then fill in the details, joining up the dots, it all falls into place quite easily.

(8) $(\mathsf{P} \land (\mathsf{Q} \lor \mathsf{R})),\ \neg((\mathsf{P} \land \mathsf{Q}) \land \mathsf{S}),\ (\neg(\mathsf{P} \land \mathsf{S}) \lor \neg\mathsf{R})\ \therefore\ \neg\mathsf{S}$

If you struggled with this, don't worry. But let's make a start: we have three premisses, and the first is a conjunction we can immediately unpack:

$(\mathsf{P} \land (\mathsf{Q} \lor \mathsf{R}))$	(Prem)
$\neg((\mathsf{P} \land \mathsf{Q}) \land \mathsf{S})$	(Prem)
$(\neg(\mathsf{P} \land \mathsf{S}) \lor \neg\mathsf{R})$	(Prem)
P	($\land$E)
$(\mathsf{Q} \lor \mathsf{R})$	($\land$E)

So now the obvious way to try to proceed is via an argument by cases using that simple disjunction, aiming for an argument with this shape:

Q	(Supp)
$\vdots$	
$\neg\mathsf{S}$	
R	(Supp)
$\vdots$	
$\neg\mathsf{S}$	
$\neg\mathsf{S}$	($\lor$E, using $(\mathsf{Q} \lor \mathsf{R})$ and the two subproofs)

And now immediately things are looking more promising, as we have two simpler subproofs to tackle.

How are we going to fill out the first subproof? We want to show $\neg\mathsf{S}$ so assume the opposite and aim for a contradiction. And indeed a contradiction is almost immediate, because if we assume S, given that we also have P and Q available, we get a conflict with the second premiss. Which gets us to

(1)	$(\mathsf{P} \land (\mathsf{Q} \lor \mathsf{R}))$	(Prem)
(2)	$\neg((\mathsf{P} \land \mathsf{Q}) \land \mathsf{S})$	(Prem)
(3)	$(\neg(\mathsf{P} \land \mathsf{S}) \lor \neg\mathsf{R})$	(Prem)
(4)	P	($\land$E 1)
(5)	$(\mathsf{Q} \lor \mathsf{R})$	($\land$E 1)
(6)	Q	(Supp)
(7)	S	(Supp)
(8)	$(\mathsf{P} \land \mathsf{Q})$	($\land$I 4, 6)
(9)	$((\mathsf{P} \land \mathsf{Q}) \land \mathsf{S})$	($\land$I 8, 7)
(10)	$\bot$	(Abs 9, 2)
(11)	$\neg\mathsf{S}$	(RAA 7–10)
(12)	R	(Supp)
(13)	$\vdots$	
(14)	$\neg\mathsf{S}$	
(15)	$\neg\mathsf{S}$	($\lor$E 5, 6–11, 12–14)

So now it remains to fill in the second subproof. We'll have to use the disjunctive third premiss, so we will need another proof by cases, with two new subproofs nested inside the currently unfinished subproof, one beginning $\neg(\mathsf{P} \wedge \mathsf{S})$, the other beginning $\neg\mathsf{R}$, and both aiming to conclude $\neg\mathsf{S}$.

But this isn't too hard; the new subproofs almost write themselves after a moment's reflection. For the first, just assume S and use reductio. For the second, we note that our new supposition contradicts something already available so we can get what we want by (EFQ).

(1)	$(\mathsf{P} \wedge (\mathsf{Q} \vee \mathsf{R}))$	(Prem)
(2)	$\neg((\mathsf{P} \wedge \mathsf{Q}) \wedge \mathsf{S})$	(Prem)
(3)	$(\neg(\mathsf{P} \wedge \mathsf{S}) \vee \neg\mathsf{R})$	(Prem)
(4)	P	($\wedge$E 1)
(5)	$(\mathsf{Q} \vee \mathsf{R})$	($\wedge$E 1)
(6)	$\quad \mathsf{Q}$	(Supp)
(7)	$\quad\quad \mathsf{S}$	(Supp)
(8)	$\quad\quad (\mathsf{P} \wedge \mathsf{Q})$	($\wedge$I 4, 6)
(9)	$\quad\quad ((\mathsf{P} \wedge \mathsf{Q}) \wedge \mathsf{S})$	($\wedge$I 8, 7)
(10)	$\quad\quad \bot$	(Abs 9, 2)
(11)	$\quad \neg\mathsf{S}$	(RAA 7–10)
(12)	$\quad \mathsf{R}$	(Supp)
(13)	$\quad\quad \neg(\mathsf{P} \wedge \mathsf{S})$	(Supp)
(14)	$\quad\quad\quad \mathsf{S}$	(Supp)
(15)	$\quad\quad\quad (\mathsf{P} \wedge \mathsf{S})$	($\wedge$I 4, 14)
(16)	$\quad\quad\quad \bot$	(Abs 15, 13)
(17)	$\quad\quad \neg\mathsf{S}$	(RAA 14–16)
(18)	$\quad\quad \neg\mathsf{R}$	(Supp)
(19)	$\quad\quad \bot$	(Abs 12, 18)
(20)	$\quad\quad \neg\mathsf{S}$	(EFQ 19)
(21)	$\quad \neg\mathsf{S}$	($\vee$E 3, 13–17, 18–20)
(22)	$\neg\mathsf{S}$	($\vee$E 5, 6–11, 12–21)

Breaking down our overall proof-task into smaller steps, this wasn't *too* horrible.

(b*) *Revisit Exercises 20(b*). Let S be the proof system with our rules for conjunction, negation and now disjunction as well.*

(1) Do results (1) to (4) from those previous exercises still obtain now we have revised what counts as the proof system S?

Straightforwardly, yes. We just need to note that the arguments for results (1) to (4) depended only on the way that the negation and conjunction rules work, and the same rules are still in place.

Use similar arguments to those outlined in those previous exercises to show:

(2) If the wffs Γ are S-consistent and $(\alpha \vee \beta)$ is one of those wffs, then either Γ, α or Γ, β (or both) are also S-consistent.

(3) If the wffs Γ are S-consistent and $\neg(\alpha \vee \beta)$ is one of those wffs, then $\Gamma, \neg\alpha, \neg\beta$ are also S-consistent.

Suppose Γ, α and Γ, β are both S-inconsistent. Then Γ proves both $\neg\alpha$ and $\neg\beta$. If $(\alpha \vee \beta)$ is also already in Γ, then Γ is S-inconsistent, because we could argue

$$
\begin{array}{l|lll}
 & \Gamma & & \text{(Premisses)} \\
\hline
 & (\alpha \vee \beta) & & \text{(iterating a premiss)} \\
 & \vdots & & \\
 & \neg\alpha & & \text{(provable from } \Gamma\text{, by our assumption)} \\
 & \neg\beta & & \text{(provable from } \Gamma\text{, by our assumption)} \\
 & & \alpha & \text{(Supp)} \\
 & & \bot & \\
 & & \beta & \text{(Supp)} \\
 & & \bot & \\
 & \bot & & (\wedge\text{E})
\end{array}
$$

In short, then, if Γ, α and Γ, β are both S-inconsistent. and $(\alpha \vee \beta)$ is in Γ, then Γ is S-inconsistent. Re-arranging that gives (2).

Now suppose $\neg(\alpha \vee \beta)$ is one the wffs Γ, and also $\Gamma, \neg\alpha, \neg\beta$ are S-inconsistent. Then we have the proof

$$
\begin{array}{l|lll}
 & \Gamma & & \text{(Premisses)} \\
\hline
 & \neg(\alpha \vee \beta) & & \text{(iterating a premiss)} \\
 & & \alpha & \text{(Supp)} \\
 & & (\alpha \vee \beta) & (\vee\ \text{I}) \\
 & & \bot & \\
 & \neg\alpha & & \text{(RAA)} \\
 & & \beta & \text{(Supp)} \\
 & & (\alpha \vee \beta) & (\vee\ \text{I}) \\
 & & \bot & \\
 & \neg\beta & & \text{(RAA)} \\
 & \vdots & & \\
 & \bot & & \text{(by our assumption that we can get from } \Gamma, \neg\alpha, \neg\beta \text{ to } \bot)
\end{array}
$$

In short, then, if $\Gamma, \neg\alpha, \neg\beta$ are S-inconsistent and $\neg(\alpha \vee \beta)$ is in Γ, then Γ is inconsistent. Re-arranging gives us (3).