Peter Smith, *Introduction to Formal Logic* (CUP, 2nd edition)

Exercises 28: A first QL language

In the language QL_1, the *proper names* are just

m: Socrates,
n: Plato,
o: Aristotle;

and the *predicates* are just

F: ① is a philosopher,
G: ① is a logician,
H: ① is wise,
L: ① loves ②,
M: ① is a pupil of ②,
R: ① prefers ② to ③.

The domain of quantification for QL_1: people, past and present.

Which of the following expressions are wffs of the language QL_1? In each case, give one possible construction history for those expressions which are wffs, and indicate which logical operators (connectives or quantifiers) are in the scope of which other logical operators. What do those expressions which are wffs mean?

(1) $(\mathsf{Lnn} \land (\mathsf{Lmn} \to \mathsf{Lnm}))$

Obviously a wff. A possible constructional history – set out in unnecessarily tedious detail! –

1. Lnn is a wff	(It's an atomic wff!)
2. Lmn is a wff	(It's an atomic wff!)
3. Lnm is a wff	(It's an atomic wff!)
4. $(\mathsf{Lmn} \to \mathsf{Lnm})$ is a wff	(From 2 and 3, by rule for forming conditionals)
5. $(\mathsf{Lnn} \land (\mathsf{Lmn} \to \mathsf{Lnm}))$ is a wff	(From 1 and 4, by rule for forming conjunctions)

So '$\to$' coming before '$\land$' in the constructional history, the conditional is inside the scope of (has narrower scope than) the conjunction.

Translation: Plato loves himself and, and if Socrates loves him then he loves Socrates too.

(2) $\forall\mathsf{x}(\mathsf{Fx} \to \mathsf{Lxm})$

Obviously a wff. A possible constructional history:

1. Fn is a wff	(It's an atomic wff!)
2. Lnm is a wff	(It's an atomic wff!)
3. $(\mathsf{Fn} \to \mathsf{Lnm})$ is a wff	(From 1 and 2, by rule for forming conditionals)
4. $\forall\mathsf{x}(\mathsf{Fx} \to \mathsf{Lxm})$ is a wff	(From 3, by rule for forming universal quantification, quantifying on n)

So '$\to$' coming before '$\forall$' in the constructional history, the conditional is inside the scope of (has narrower scope than) the universal quantification.

Translation: Everyone is such that, if they are wise, they loves Socrates – i.e. everyone wise loves Socrates.

(3) $(\mathsf{Go} \land \neg\exists\mathsf{x}(\mathsf{Gx} \land \mathsf{Hx}))$

Another wff. A possible constructional history:

1. Go is a wff	(It's an atomic wff!)
2. Ho is a wff	(It's an atomic wff!)
3. $(\mathsf{Go} \land \mathsf{Ho})$ is a wff	(From 1 and 2, by rule for forming conjunctions)

4. $\exists x(Gx \wedge Hx)$ is a wff — (From 3, by rule for forming existential quantification, quantifying on o)
5. $\neg\exists x(Gx \wedge Hx)$ is a wff — (From 4, by rule for forming negation)
6. $(Go \wedge \neg\exists x(Gx \wedge Hx))$ — (From 1 and 5 by rule for forming conjunctions)

[We could of course alternatively have introduced a separate wff e.g. Gn, conjoined that with Hn to form $(Gn \wedge Hn)$, and then existentially quantified *that*.]

So the right-hand occurrence of '$\wedge$' is introduced before, is in the scope of, '$\exists$'; and both are in the scope of the negation; and those three operators are all in the scope of the left-hand occurrence of '$\wedge$'.

Translation: Aristotle is a logician but no logician is wise. [What a silly claim!]

(4) $\forall x(Fx \to Gx \wedge Hx)$

Not a wff! It's missing some brackets. $\forall x(Fx \to (Gx \wedge Hx))$ *would* be a wff – saying that every philosopher is a wise logician [which would be another very silly claim!]

(5) $(\exists xFx \vee \exists xGx)$

A wff. A possible constructional history:

1. Fn is a wff — (It's an atomic wff!)
2. Gn is a wff — (It's an atomic wff!)
3. $\exists xFx$ is a wff — (From 1, by rule for forming existential quantification, quantifying on n)
4. $\exists xGx$ is a wff — (From 2, by rule for forming existential quantification, quantifying on n)
5. $(\exists xFx \vee \exists xGx)$ is a wff — (From 3 and 4, by rule for disjunctions)

We could of course have started with different names at lines 1 and 2! Both the occurrences of '$\exists$' are in the scope of '$\vee$'.

Note: a wff *can* have two occurrences of quantifiers using the same variable. But (in our syntax) one occurrence cannot be in the scope of another. That's the case here: if you put our constructional history in tree form (or draw it the other way up as a sort of parse tree) the two occurrences of $\exists x$ will be on different branches — neither is in the scope of the other.

Translation: Either there's at least one philosopher or there's at least one logician.

(6) $\exists x(Fx \vee \exists xGx)$

In our syntax, not a wff. We can form e.g. the wff $(Fn \vee \exists xGx)$. But if we want to existentially quantify on n here, we need to choose a new variable, not already in the wff, as in $\exists y(Fy \vee \exists xGx)$. [Note, some texts set up the syntax so that $\exists x(Fx \vee \exists xGx)$ does count as a wff: this is one of those minor differences between treatments that doesn't really matter too much – but our rule *you always quantify using a variable new to the wff* is a sort of safety-first rule that makes it easier to avoid some potential pitfalls.]

(7) $(Fx \vee \exists yGy)$

Not a wff. There's a dangling, unquantified variable.

(8) $\exists x(Fx \vee \exists yGy)$

A wff. A possible constructional history:

1. Fn is a wff — (It's an atomic wff!)
2. Gn is a wff — (It's an atomic wff!)
3. $\exists yGy$ is a wff — (From 2, by rule for forming existential quantification, quantifying on n)
4. $(Fn \vee \exists yGy)$ is a wff — (From 1 and 2, by rule for disjunctions)

5. $\exists x(Fx \lor \exists yGy)$	(From 4, by rule for forming existential quantification, quantifying on n)

[Of course we could have used different names in 1 and 2.] The second quantifier is in the scope of the disjunction. Both are in the scope of the first quantifier.

Translation: There's someone who is a philosopher unless there is a logician.

(9) $\forall y \exists x\, Rxyx$

A wff. A possible constructional history:

1. $Rmnm$ is a wff	(It's an atomic wff!)
2. $\exists x Rxnx$ is a wff	(From 1, by rule for forming existential quantification, quantifying on m)
3. $\forall y \exists x Rxyx$ is a wff	(From 2, by rule for forming universal quantification, quantifying on n)

The existential quantification is of course inside the scope of the universal quantification.

Translation: Everyone prefers someone to themself.

(10) $\exists x \forall y \exists x\, Rxyx$

Not a wff (two quantifiers using the same variable, one inside the scope of the other).

(11) $(Lmn \land \forall x(Lmx \to Lxn))$

A wff. A possible constructional history:

1. Lmn is a wff	(It's an atomic wff!)
2. Lmo is a wff	(It's an atomic wff!)
3. Lon is a wff	(It's an atomic wff!)
4. $(Lmo \to Lon)$ is a wff	(From 2 and 3, by rule for forming conditionals)
5. $\forall x(Lmx \to Lxn)$	(From 4, by rule for forming universal quantification, quantifying on m)
6. $(Lmn \land \forall x(Lmx \to Lxn))$ is a wff	(From 1, 5, by rule for forming conjunctions)

The conditional is inside the scope of the universal quantification, and both are in the scope of the conjunction.

Translation: Socrates loves Plato, and whoever Socrates loves also loves Plato.

(12) $(Gn \to \exists z(Fz \land Lnz))$

A wff. A possible constructional history:

1. Gn is a wff	(It's an atomic wff!)
2. Fm is a wff	(It's an atomic wff!)
3. Lnm is a wff	(It's an atomic wff!)
4. $(Fm \land Lnm)$ is a wff	(From 2 and 3, by rule for forming conjunctions)
5. $\exists z(Fz \land Lnz)$	(From 4, by rule for forming existential quantification, quantifying on o)
6. $(Gn \to \exists z(Fz \land Lnz))$ is a wff	(From 1, 5, by rule for forming conditionals)

The conjunction is inside the scope of the existential quantification, and both are in the scope of the conditional.

Translation: If Plato is a logician, then he loves some philosopher.

(13) $\forall y(Gy \to \exists z(Fz \land Lyz))$

A wff – just universally quantify the previous wff on n (and now the other logical operators are all inside the scope of the new initial quantifier). So this wff says about everyone what the previous one says about Plato. It says, everyone is such that, if they are a logician, they love some philosopher. Or better,

Translation: Every logician loves some philosopher.

(14) $\neg\exists y(Fy \land \forall x Rxoy)$

A wff. A possible constructional history:

1.	Fn is a wff	(It's an atomic wff!)
2.	$Rmon$ is a wff	(It's an atomic wff!)
3.	$\forall x Rxon$ is a wff	(From 2, by rule for forming universal quantification, quantifying on m)
4.	$(Fn \land \forall x Rxon)$ is a wff	(From 1, 3, by rule for forming conjunctions)
5.	$\exists y(Fy \land \forall x Rxoy)$ is a wff	(From 4, by rule for forming existential quantification, quantifying on n)
6.	$\neg\exists y(Fy \land \forall x Rxoy)$	(From 5 by rule for forming negations)

The universal quantifier is inside the scope of the conjunction , and both are in the scope of the existential quantifier; and all three of those operators are inside the scope of the initial negation. Clunkily it says: it isn't the case that someone if a philosopher such that everyone prefers Aristotle to them.

Slightly smoother translation: No philosopher is such that *everyone* prefers Aristotle to them.