The MRDP Theorem*

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September 12, 2024

Here is Hilbert is his famous address of 1900:

> The supply of problems in mathematics is inexhaustible, and as soon as one problem is solved numerous others come forth in its place. Permit me in the following, tentatively as it were, to mention particular definite problems, drawn from various branches of mathematics, from the discussion of which an advancement of science may be expected.

He goes on to highlight no less than 23 problems. The first, of course, is Cantor's problem of the cardinal number of the continuum. The tenth, not as profound but still of considerable interest to logicians, is this:

> Given a diophantine equation with any number of unknown quantities and with rational integral numerical coefficients: to devise a process according to which it can be determined by a finite number of operations whether the equation is solvable in rational integers.

I'll explain what a diophantine equation is in just a moment. Talk of a devising a process for determining something in a finite number of operations we'll take as a gesture towards a request for an algorithm (but of course, Hilbert in 1900 hadn't got the modern, post 1930, conception of a computable procedure).

Hilbert's Tenth Problem took seventy years to resolve. Important work towards a solution was done by Julia Robinson and Martin Davis, with a contribution from Hilary Putnam. The final, and key, step however was made by a young Russian mathematician, Yuri Matiyasevich. Putting everything together, we get the MRDP theorem, settling the Tenth Problem in the negative: provably, there is *no* algorithmic way of determining whether some arbitrary diophantine equation has a solution.

I'm not going to do say more than a sentence or two about the proof of the key step in establishing the MRDP theorem. My aim in this talk, rather, is to explain in broad terms what the result amounts to – and explain too why it is indeed of interest to logicians.

1 Diophantine equations and diophantine sets

I obviously need to begin by quickly explaining what is meant by a diophantine equation. Well, start with the idea of a polynomial:

*This is an only slightly tidied version of handout which I dashed off back in the day for an informal short talk I gave to a group of philosophers, mostly grad. students who are technically ept but not mathematicians. Hence the introductory level, the reminders of elementary mathematical ideas and results, and the way the results are spun. The usual warning applies: just because a document is LaTeXed and so *looks* authoritative, it doesn't mean that it *is* authoritative. I don't pretend to much expertise here. For the latest version, visit http://www.logicmatters.net/igt/further-notes/. Corrections and suggestions for improvement are very welcome via peter_smith at me.com.

Definition 1.1. *A* polynomial *in the k variables $x_1, x_2, \ldots, x_k$ is a (finite) sum of expressions of the type $c_i v_1 v_2 v_3 \ldots v_j$ (or simply c_0) where the coefficients c_i are integers (positive or negative) and v_i are variables.*

Of course, we'd informally use exponential notation and abbreviate e.g. $xyyxxx$ by x^4y^2; though note that, crucially, an exponential function such as n^x is *not* polynomial, nor are such functions as $\sqrt{x}$. And rather than e.g. $(x + -2y)$ we would ordinarily write $x - 2y$. With those familiar usages, then, some examples of polynomials in x and y are: $x^2 - 4x + 3$, $x^3 - 3xy^2 + 7x - 9$, $-7x^2y^2 + 4y^2 - 17xy$, etc., etc.

Definition 1.2. *A* diophantine equation*, in traditional form, is simply an equation $p = 0$ for polynomial p.*

Then, exactly as you would expect,

Definition 1.3. *A solution of the equation $p = 0$ (in the natural numbers) is an assignment of numbers $n_1, n_2, \ldots n_k$ to the variables $x_1, x_2, \ldots, x_k$ in p such that the polynomial indeed evaluates to 0.*

A few examples:

1. The school-room quadratic equation $x^2 - 4x + 3$ has two roots, 1, 3.
2. $x^2 + y^2 - z^2 = 0$ has many solutions, the Pythagorean triples (3, 4, 5), (5, 12, 13), (7, 24, 25), (8, 15, 17), ...
3. $x^3 + y^3 - z^3 = 0$ has no solutions in non-zero integers. This is implied, needless to say, by the industrial strength Fermat's Last Theorem that there are no non-trivial solutions $x^n + y^n - z^n = 0$ for $n > 2$. But the particular case for $n = 3$ can be proved much more simply, though certainly still rather non-trivially (see http://www.mathpages.com/home/kmath009.htm).
4. Then, for a quirkier example, $x^2 - 2y^4 + 1 = 0$ has exactly two solutions (1, 1) and (239, 13), for which the proof is again non-trivial (the Norwegian mathematician Wilhelm Ljunggren did a lot of work on diophantine equations like this).

We can now pose

Hilbert's 10th Problem *Is there an algorithm which determines, of an arbitrary diophantine equation, whether it has a solution in the natural numbers?*

Actually, Hilbert originally asked about solutions in the 'rational integers', and by that he means integers positive, zero and negative: but there's an elementary proof that our formulation of the Problem here is equivalent. So from now on, talk of numbers will always mean natural numbers.

2 Linking to the familiar ...

You might think that we are already beginning to head off in a direction that is going to take us well away from the sort of territory that might be familiar to a logician who knows only the usual smidgin or two about formalized arithmetics. But not so.

Suppose we say

Definition 2.1. *A* positive polynomial *is a polynomial all of whose co-efficients are positive.*

And we'll correspondingly say

Definition 2.2. *A* positive diophantine equation *is an equation between two positive polynomials.*

Then we have a trivial

Observation 2.3. *A diophantine equation, in traditional form, is equivalent to a positive diophantine equation.*

Just shuffle negative terms across the identity sign from left to right! – for example, the diophantine equation $x^3 - 3xy^2 + 7x - 3 = 0$ is trivially equivalent to $x^3 + 7x = 3xy^2 + 3$.

And now here's another elementary

Observation 2.4. *The atomic wffs of L_A – the standard first-order language of arithmetic – express positive diophantine equations.*

By L_A I normally mean the language with non-logical vocabulary $(S, +, \times, 0)$, but for present purposes it would evidently make no odds at all if you meant instead the ring language with non-logical vocabulary $(+, \times, 0, 1)$. And why does this second observation hold? Because the only atomic wffs are those of the form $t_1 = t_2$ for two terms t_i; and the terms of L_A are in effect what you get from adding and multiplying (positive!) numerals and variables – i.e. express positive polynomials.

So: the claim (a) that a (positive) diophantine equation has a solution is equivalent to (b) the claim that the existential closure of an atomic wff of L_A is true. And the question whether there is way of deciding claims of type (a) is the question whether there is a way of deciding the truth of claims of type (b). And – in *this* guise – Hilbert's 10th Problem doesn't look at all remote from the familiar sort of question that we, as logicians, might already care about.

3 Diophantine sets

Back to informal maths for a bit. Let $p(x, y_1, y_2, \ldots, y_k)$ represent a polynomial with the variables as displayed. Then

Definition 3.1. *The set of numbers K is diophantine if and only if there is a polynomial p such that $x \in K$ iff there are values of y_i such that $p(x, y_1, y_2, \ldots, y_k) = 0$.*[1]

A few examples:

1. The set of composite (non-prime) numbers is diophantine. Consider the simple polynomial $x - (y_1 + 2)(y_2 + 2)$; this takes a zero for each x which is the multiple of two numbers greater than one.

2. The set of numbers which are either multiples of two or multiples of three is diophantine. Evidently, $x - 2y$ takes a zero for each x which is even; and $x - 3y$ takes a zero for each x which is a multiple of three. So $(x - 2y)(x - 3y)$ takes a zero for each x which is either a multiple of two or a multiple of three.

[1] We can obviously generalize to define n-ary diophantine relations-in-extension: The K set of n-tuples of numbers is diophantine if and only if there is a polynomial p such that $(x_1, x_2, \ldots, x_n) \in K$ iff there are values of y_i such that $p(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_k) = 0$. But we won't need to get general here.

3. For a trickier case, the set of numbers which are not powers of two is also diophantine. What we want to say is that x is in this set if there is a number y_1 such that (i) y_1 divides x, and (ii) $y_1 > 1$ but (iii) 2 doesn't divide y_1. For condition (i), we need there to be a y_2 such that $x - y_1y_2$ is zero. For condition (ii), we need there to be a y_3 such that $y_1 - y_3 - 2$ is zero. For condition (iii), we need there to be a y_4 such that $y_1 - 2y_4 - 1$ is zero. And how do we force three different conditions all to be zero? By requiring the sum of their squares to be zero! So consider the polynomial
$$(x - y_1y_2)^2 + (y_1 - y_3 - 2)^2 + (y_1 - 2y_4 - 1^2)$$
This will hit a zero for any x which isn't a pure power of two, as required.

The second and third examples show, in effect how we can disjoin or conjoin diophantine conditions to get more and more complex ones. This sort of trick enables us to get quite a way, by trial and error. Thus, with some effort, we can show that

4. The set of prime numbers is diophantine. Take the polynomial $(wz+h+j-q)^2 + ((gk+2g+k+1)(h+j)+h-z)^2+(16(k+1)^3(k+2)(n+1)^2+1-f^2)^2+(2n+p+q+z-e)^2+(e^3(e+2)(a+1)^2+1-o^2)^2+((a^2-1)y^2+1-x^2)^2+(16r^2y^4(a^2-1)+1-u^2)^2+(n+l+v-y)^2+((a^2-1)l^2+1-m^2)^2+(ai+k+1-l-i)^2+(((a+u^2(u^2-a))^2-1)(n+4dy)^2+1-(x+cu)^2)^2+(p+l(a-n-1)+b(2an+2a-n^2-2n-2)-m)^2+(q+y(a-p-1)+s(2ap+2a-p^2-2p-2)-x)^2+(z+pl(a-p)+t(2ap-p^2-1)-pm)^2$. Then $k+2$ is prime so long as there are values of the other variables for which the polynomial equals zero.

(The proof of that claim can be safely left as an exercise . . .)

OK. The diophantine specification of even very familiar and elementarily-definable sets of numbers looks as if it soon can get horribly complicated. So how far can we go? Is there any other (perhaps more familiar) way of alternatively characterizing the diophantine sets?

4 Very quick reminders

At this point let's pause for some very speedy reminders of notions from the theory of recursive (i.e. computable) functions.

Definition 4.1. *A set of numbers K is* recursively enumerable *iff it is (i) the range of a total recursive function – or equivalently, it is (ii) the domain of a partial recursive function.*

What does this come to? Definition (i) says that K is recursively enumerable if there is an algorithmically computable (total) function f such that as you go through $f(0), f(1), f(2), \ldots$ you spit out values $k_0, k_1, k_2 \ldots$ in K, with any member of K eventually appearing (perhaps with repetitions). The equivalent definition (ii) says that there is an algorithm such that the set of input numbers for which the algorithm halts is exactly the set of numbers in K. [Reality check: why are these equivalent?]

Definition 4.2. *A set of numbers K is* recursive *iff there is a recursive function $f\colon K \to \{0,1\}$, such that $n \in K \leftrightarrow f(n) = 1$.*

Think of 0, 1 as truth-values: then our definitions says that there is an algorithm for deciding membership of K.

A basic result is that

Theorem 4.3. *If a set K and its complement $N - K$ are both r.e., then K is recursive.*

(Just start enumerating K and its complement $N - K$ in parallel, and any number will turn up one way or the other, thus deciding whether it is in K.)

But the key theorem we now need to recall is

Theorem 4.4. *There are sets of numbers which are r.e. but not recursive.*

For example, take the set of Gödel numbers of theorems of first-order PA. This is recursively enumerable (for we can start mechanically spitting out all the theorems one after another and take their codes – *IGT*, Thm 30.5), but it is not recursive (there is no mechanical means of deciding what's a PA theorem – *IGT*, Thm 30.3).

5 And now, the MRDP theorem

Back to the question about characterizing diophantine sets. An immediate first theorem is

Theorem 5.1. *Any diophantine set of numbers is recursively enumerable.*

Proof. Suppose that K is determined by the diophantine equation $p(x, y_1, y_2, \ldots, y_k) = 0$. Churn through all the $k+1$-tuples of possible values for the variables $x, y_1, y_2, \ldots, y_k$ in some zig-zag order, and whenever p evaluates to zero for a given $k+1$-tuple as input, write down the value of x. That is plainly a mechanical computation, and by definition, we will eventually generate any member of K as we go along. □

That was easy. But what Matiyasevich finally established is the vastly more difficult converse:

Theorem 5.2. *Any recursively enumerable set of numbers is diophantine.*

How is this proved? With ingenuity! It was known from Martin Davis, Hilary Putnam and Julia Robinson's work that every recursively enumerable set is 'exponential diophantine' (the idea is that we allow exponentials as well as sums and products in our definition of a suped-up diophantine equation).[2] So if we can show that the exponential can be defined in a diophantine way then this will complete the proof. Matiyasevich did the trick with clever use of Fibonacci numbers which tend to an exponential growth-rate.[3]

So, what are the implications of Theorem 5.2?

Let K be a diophantine set, so there is a polynomial p such that $x \in K$ if and only if $\exists y_1 \exists y_2 \ldots \exists y_k \, p(x, y_1, y_2, \ldots, y_k) = 0$. Now suppose there is a decision procedure for determining whether a diophantine equation $p(x, y_1, y_2, \ldots, y_k) = 0$ is satisfied. Then, fixing on the value of x, we can apply that decision procedure to determine whether $x \in K$. But that means that the supposition that there is a decision procedure for determining whether a polynomial hits zero implies that any recursively enumerable set K, being diophantine, is decidable. But we know from Theorem 4.3 that that's false. Whence

[2] Davis and Putnam proved the result, assuming a conjecture in number theory: Robinson discovered how to avoid the conjecture and simplified the proof. The result is their joint paper 'The Decision Problem for Exponential Diophantine Equations', *The Annals of Mathematics* Vol. 74, No. 3, (Nov., 1961), pp. 425–436. Much later, J. P. Jones and Matiyasevich found a very much neater and intuitive proof, 'Register Machine Proof of the Theorem on Exponential Diophantine Representation of Enumerable Sets', *The Journal of Symbolic Logic*, Vol. 49, No. 3 (Sep., 1984), pp. 818–829.

[3] For some details of how Matiyasevich pulled off the trick, or rather of a simplified version of his proof, see e.g. Martin Davis's 'Hilbert's Tenth Problem is Unsolvable', *American Mathematical Monthly*, Vol. 80, No. 3 (Mar., 1973), pp. 233–269.

Theorem 5.3 (The MRDP theorem)**.** *There is no algorithmic decision procedure for determining whether an arbitrary diophantine equation has a solution.*

6 'PA can prove Matiyasevich's theorem'

Some more reminders.

Definition 6.1. *An L_A wff of the form $\exists \mathsf{y}_1 \exists \mathsf{y}_2, \ldots \exists \mathsf{y}_\mathsf{k} \varphi(\mathsf{x}, \mathsf{y}_1, \mathsf{y}_2, \ldots, \mathsf{y}_\mathsf{k})$ where φ contains at most bounded quantifiers, is said to be a (one-place open) Σ_1 wff.*

(We'll henceforth use standard shorthand to gather variables and blocks of quantifiers together, and write e.g. $\exists \vec{\mathsf{y}} \varphi(\mathsf{x}, \vec{\mathsf{y}})$ for short.) Then we have the following result:

Theorem 6.2. *A set of numbers K is r.e. just in case, for some (one-place open) Σ_1 wff $\psi(\mathsf{x})$, $n \in K$ iff $\mathcal{N} \vDash \psi(\overline{\mathsf{n}})$, where $\mathcal{N}$ is the standard model, and $\overline{\mathsf{n}}$ is L_A's standard numeral for n.*

Proof. Any partial recursive function $f(x) = u$ can be expressed using some two-place open Σ_1 wff $\exists \vec{\mathsf{y}} \varphi(\mathsf{x}, \mathsf{u}, \vec{\mathsf{y}})$, by standard constructions. So then the corresponding one-place open Σ_1 wff $\exists \mathsf{u} \exists \vec{\mathsf{y}} \varphi(\mathsf{x}, \mathsf{u}, \vec{\mathsf{y}})$ will be satisfied by those arguments for which f has a value. Now appeal to version (ii) of Defn. 4.1. □

Let's give another companion definition, motivated by our earlier observations about positive diophantine equations:

Definition 6.3. *An L_A wff of the form $\exists \vec{\mathsf{y}}(\tau_1(\mathsf{x}, \vec{\mathsf{y}}) = \tau_2(\mathsf{x}, \vec{\mathsf{y}}))$, where τ_i are terms, will be said to be (one-place open) diophantine.*

Then of course, we have

Observation 6.4. *A set of numbers K is diophantine just in case, for some (one-place open) diophantine wff $\psi(\mathsf{x})$, $n \in K$ iff $\mathcal{N} \vDash \psi(\overline{\mathsf{n}})$.*

Now, Matiyasevich's Theorem 5.2 says, in effect, that whatever set is the extension of some (one-place, open) Σ_1 formula is the extension of some diophantine formula. And, in a sense, PA can prove that. That is to say, we have

Theorem 6.5. *For every (one-place, open) Σ_1 formula $\psi(\mathsf{x})$ of L_A there is a diophantine formula $\delta(\mathsf{x})$ such that $PA \vdash \forall \mathsf{x}(\psi(\mathsf{x}) \leftrightarrow \delta(\mathsf{x}))$.*[4]

7 MRDP and Gödelian incompleteness

Note an interesting corollary of our last theorem, when we insert negations on either side of the biconditional. Suppose $\varphi(\mathsf{x})$ is some Π_1 sentence of PA. Then, there will be a wff $\forall \vec{\mathsf{y}}(\tau_1(\mathsf{x}, \vec{\mathsf{y}}) \neq \tau_2(\mathsf{x}, \vec{\mathsf{y}}))$ such that PA $\vdash \forall \mathsf{x}(\varphi(\mathsf{x}) \leftrightarrow \forall \vec{\mathsf{y}}(\tau_1(\mathsf{x}, \vec{\mathsf{y}}) \neq \tau_2(\mathsf{x}, \vec{\mathsf{y}})))$. Now recall how a standard Gödel sentence G is constructed for PA; we plug a numeral into a certain Π_1 open sentence. So this means that

[4] In fact, this result can be sharply improved: $I\Sigma_1$, the theory with induction just for Σ_1 wffs, is already more than enough.

By the way, our theorem 6.5 looks different from e.g. Kaye's Result 7.8 on p.88 of his *Models of Peano Arithmetic*. But there is an easy result that, even in PA^-, an $\exists_1$ wff – i.e. an existential quantification of a quantifier-free kernel – is equivalent to a diophantine wff. Hint: just put the $\exists_1$ wff's kernel into DNF; the negated atoms of the form $\tau_1 \neq \tau_2$ are equivalent to disjunctions of the form $\tau_1 < \tau_2 \vee \tau_1 > \tau_2$ with both inequalities being diophantine; and we know how to deal with disjunctions and conjunctions of diophantine conditions.

Theorem 7.1. *If G is a standard Π_1 Gödel sentence of PA, then there is a corresponding sentence to the effect that a certain diophantine equation has no solutions which is also undecidable in PA.*

Which makes a first connection with undecidability results. (The result generalizes, in fact, to any Π_1 undecidable sentence.)

But now let's tackle incompleteness more directly, without relying on a prior proof of Gödel's First Theorem: we'll re-prove incompleteness from the MRDP theorem. Begin be noting that if T is a formal theory which e.g. includes Robinson arithmetic, then,

Observation 7.2. *If the positive diophantine equation $\tau_1(\vec{x}) = \tau_2(\vec{x})$ has the solution $\vec{n}$, then $T \vdash \tau_1(\vec{n}) = \tau_2(\vec{n})$.*

(for Robinson Arithmetic correctly decides every quantifier-free sentence).

Now consider this theory T's theorems that, in effect, say that some diophantine equation lacks a solution. These are the theorems of the form $\forall \vec{x}\, \tau_1(\vec{x}) \neq \tau_2(\vec{x})$. Then we have

Theorem 7.3. *If T is consistent, and $T \vdash \forall \vec{x}\, \tau_1(\vec{x}) \neq \tau_2(\vec{x})$, then the theorem is true.*

Proof. If $\forall \vec{x}\, \tau_1(\vec{x}) \neq \tau_2(\vec{x})$ is false, then for some $\vec{n}$, $\tau_1(\vec{n}) = \tau_2(\vec{n})$, so by our last observation and the assumption on T, $T \vdash \tau_1(\vec{n}) = \tau_2(\vec{n})$ and is inconsistent, contrary to hypothesis. □

OK. So now consider the set D of (Gödel numbers for) the diophantine equations such that it is provable-in-T that they have no solutions. Then we have

1. D is recursively enumerable. (Just recursively enumerate the theorems of T – here we rely on the fact that we are dealing with a formal theory in the kosher sense of being recursively axiomatized. Now, as we go along, throw away all those theorems which don't have the form $\forall \vec{x}\, \tau_1(\vec{x}) \neq \tau_2(\vec{x})$. The filtering can be done mechanically, so we are left with a recursive enumeration of D.)

2. If U is the set of (Gödel numbers for) diophantine equations which really do have no solutions, then $D \subseteq U$ (by our last theorem, on this matter provability-in-T is reliable, so everything in D is indeed in U).

3. U is not recursively enumerable. (U's complement, the set of *solvable* diophantine equations *is* recursively enumerable – do a zig-zagging though possible equations and possible n-tuples of values for their variables, testing for solutions as you go. If U were r.e. too, then U would be recursive – cf Theorem 4.3. But by the MRDP theorem U isn't recursive!)

4. Hence $D \neq U$ and there are (Gödel numbers for) diophantine equations in $U - D$.

In other words, there are wffs of the particularly simple form $\forall \vec{x}\, \tau_1(\vec{x}) \neq \tau_2(\vec{x})$ which are true but unprovable in T, so long as T is consistent and contains enough arithmetic.

Could T *disprove* one of those true-but-unprovable sentences? Well, if $\forall \vec{x}\, \tau_1(\vec{x}) \neq \tau_2(\vec{x})$ is true, each $\tau_1(\vec{n}) \neq \tau_2(\vec{n})$ is true, and T can prove such quantifier-free truths. But then if $T \vdash \neg\forall \vec{x}\, \tau_1(\vec{x}) \neq \tau_2(\vec{x})$. i.e. if $T \vdash \exists \vec{x}\, \tau_1(\vec{x}) = \tau_2(\vec{x})$, T would be ω-inconsistent.

Hence we get

Theorem 7.4 (Gödel's First Incompleteness Theorem)**.** *If T is an ω-consistent (and hence consistent) formal theory which contains enough arithmetic, then there are wffs of the form $\forall \vec{x}\, \tau_1(\vec{x}) \neq \tau_2(\vec{x})$, which say that certain diophantine equations have no solution, which are true but which are formally undecidable in T.*

Why is this new proof of the old result interesting? Well, first the undecidable sentences are of an attractively natural kind, mathematically speaking. (To qualify that a bit: claims that certain diophantine equations have no solution are certainly natural enough. Of course, the particular instances of the equations that may be involved in the undecidable cases may be rather 'unnaturally' complicated. Still, it remains quite plausible to say – albeit rather arm-wavingly – that we are in more mainstream territory than with 'classical' undecidable Gödel sentences.) Second, relatedly, while *diagonalization* is in the background – in particular, in establishing Theorem 4.3 – the proof doesn't seem to depend at any stage on *self-referential* tricks.

8 A final teaser ...

There's more to be said. But as a teaser, an incitement to do some more exploration, let me finish for today by just stating a rather startling result in the neighbourhood, which is rooted particularly in Julia Robinson's work related to the MRDP theorem. This is from a paper by Verena Dyson, James Jones and John Sheperdson (in *Archiv Math. Logik* 22 (1982) 51–60).

Theorem 8.1. *Let T be any axiomatizable ω-consistent theory containing Robinson Arithmetic. Then there is an n (different for different theories) such that the following sentence is undecidable in T:*

$$\begin{aligned}&\exists a\exists b\forall(i \leq \overline{n})\exists s\exists w\exists p\exists q\forall j\forall v\exists e\exists g\\ &\quad\{(s+w)^2+3w+s=2i\wedge([j=w\wedge v=q]\vee[j=3i\wedge v=p+q]\\ &\qquad\vee[j=s\wedge(v=p\vee(i=\overline{n}\wedge v=q+\overline{n}))]\vee[j=3i+l\wedge v=pq]\\ &\qquad\quad\rightarrow a=v+e+ejb\wedge v+g=jb)\}.\end{aligned}$$

One can but gasp in wonderment ...[5]

[5] In his nice book aimed at undergraduates *Set Theory, Logic and Their Limitations* (CUP, 1996), Moshé Machover explains the content of the MRDP Theorem in his short introductory chapter on 'Facts from recursion theory'. He doesn't prove the Theorem, but in his next chapter takes it for granted in using it to neatly prove some results about undecidability and incompleteness.

If you want more information about how the Theorem is proved, there are accounts in e.g. Bell and Machover's *A Course in In Mathematical Logic* (North-Holland, 1977) and Hodel's *An Introduction to Mathematical Logic* (PWS 1995, Dover reprint 2013). But most accessibly, with interesting related material, there is a whole book on *Hilbert's Tenth Problem* in the AMS Student Mathematical Library by Murty and Fodden (AMS, 2019).