Solutions: Functions

1. First, some very elementary reality checks.
   (a) Which of these definitions is legitimate?
      i. f is a function from domain $\mathbb{N}$ to codomain $\mathbb{N}$ defined by $f(n) = n + 1$. Or, in an obvious shorthand: $f : \mathbb{N} \to \mathbb{N}$ where $f(n) = n + 1$.
      ii. $f : \mathbb{N} \to \mathbb{N}$ where $f(n) = n + 1\frac{1}{2}$.
      iii. $f : \mathbb{N} \to \mathbb{N}$ where $f(n) = n - 1$.
      iv. $f : \mathbb{Z} \to \mathbb{Z}$ where $f(n) = n - 1$.
      v. $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = \sqrt{x}$
      vi. $f : \mathbb{R}^+ \to \mathbb{R}$ where $f(x) = \sqrt{x}$
      vii. $f : \mathbb{R}^+ \to \mathbb{R}^+$ where $(f(x))^2 = x$

      i. Evidently fine, as this is just the function which takes a number to its successor

      ii. Equally evidently not fine at all. There are no fractions in the codomain $\mathbb{N}$, so the supposed definition never gives a value in the codomain to the function.

      iii. When $n = 0$, $f(n)$ lacks a value in the codomain, so again this doesn't well-define a function (one failure to assign an input value ('argument') in the domain to an output value in the codomain is enough to foul things up).

      iv. This is in order: given any integer, we can always subtract one.

      v. Negative real numbers lack real square roots, so this definition fails.

      vi. Even if we look at just positive real numbers, the problem is that they have two square roots. So unless we have stipulated that $\sqrt{x}$ uniquely denotes the positive square root of x we will be in trouble. Given that stipulation, though, this definition is then in order.

      vii. In good order.

   (b) Which of these functions is injective, which surjective, which bijective?
      i. $f : \mathbb{N} \to \mathbb{N}$ where $f(n) = 2n$.
      ii. $f : \mathbb{N} \to \mathbb{E}$ where $f(n) = 2n$.
      iii. $f : \mathbb{N} \to \mathbb{N}$ where $f(n) = n/2$ if n is even, and $f(n) = (n + 1)/2$ otherwise.
      iv. $f : \mathbb{N} \to \mathbb{N}$ where $f(n)$ is the number of grapes Julius Ceasar ate on his n-th birthday (assuming that is always a whole number)!
      v. $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = 2x$.
      vi. $f : \mathbb{R} \to \mathbb{R}^+$ where $f(x) = x^2$.

      i. Injective (different numbers are mapped to different results), not surjective (odd numbers don't get mapped to).

      ii. Injective *and* surjective, i.e. bijective. So although there is a sense in which the function here 'does the same' as the function mentioned in (i), one of them is surjective and the other isn't. We'll comment more about this sort of phenomenon below.

iii. Surjective, not injective.

iv. Not surjective (some numbers are not the numbers of grapes consumed by Caesar), not injective (eventually Caesar dies, and every post-mortem birthday is a day when the number of grapes consumed is zero)!

v. Bijective.

vi. Surjective (ever positive real has a square root, so is the square of some real), but not injective (x and $-x$ get mapped to the same value).

(c) Let A be the set $\{1, 2, 3\}$ and B be the set $\{1, 2\}$:
   i. How many functions $f\colon A \to B$ are injective, how many surjective?
   ii. How many functions $f\colon B \to A$ are injective, how many surjective?
   iii. How many functions $f\colon A \to A$ are bijective?

i. None are injective (as at least two arguments in A will have to be mapped to the same value in B). How many distinct functions $f\colon A \to B$ are there in total? $2 \times 2 \times 2$ as there are two choices for every argument in A. Only two of the these functions, $f_1(n) = 1$ for each $n \in A$, similarly $f_2(n) = 2$ for each $n \in A$, are not surjective. So the other six must be.

ii. None are surjective. And for an injective function $f\colon B \to A$, the argument 1 can be mapped to a value in B three ways, and that always leaves two other possible different values to map 2 to: so that's six combinations in all. [Question: is the fact we got the same answer for the number of surjections $f\colon A \to B$ and the number of injections $f\colon B \to A$ an accident of our example?]

iii. Again six! 1 can be mapped to any of three values, leaving two choices for 2, which fixes the value for 3.

(d) In which cases are the functions f and g mentioned one and the same?
   i. $f : \mathbb{N} \to \mathbb{N}$ where $f(n) = n(n + 1)/2$; $g : \mathbb{N} \to \mathbb{N}$ where $g(n)$ is the sum of the first n natural numbers.
   ii. $f : \mathbb{N} \to \mathbb{N}$ where $f(n) = n^2$; $g : \mathbb{Z} \to \mathbb{Z}$ where $g(n) = n^2$.
   iii. $f : \mathbb{N} \to \mathbb{N}$ where $f(n) = 2n$; $g : \mathbb{N} \to \mathbb{E}$ where $g(n) = 2n$.

i. The same function.

ii. We have only said in the preamble what it is for functions $f\colon A \to B$ and $g\colon A \to B$, with the same domains and codomains, to be identical. We said, in this case, functions are the same if they match arguments to values in the same way.

How do we extend this to the case where domains are different? The natural thing to say is that, as we expand a domain (from $\mathbb{N}$ to $\mathbb{Z}$, say) we get more matching pairs $\langle$*argument, corresponding value*$\rangle$, so – identifying functions by the matches they set up – we get different functions. In a slogan, then: "different domains, different functions". (An aside. If we think of functions intensionally, as *rules*, then we might be tempted to say that the same rule can be applied to different domains, so squaring natural numbers and squaring integers (positive and negative) involves the same function. But on second thoughts, is it really exactly the same rule – after all, recall how we need to be taught how to extend multiplication from naturals to signed integers. In moving to a different domain, we are also extending (so changing) the functional rule. So after all, perhaps we would still need to indeed say "different domains, different functions", strictly understood.)

iii. The rule "different domains, different functions" doesn't settle this case, though, where only the codomains differ (there is unnecessary stuff in one codomain, not the other). There are two ways to go here:

1. Our extensional line on functions might be thought to be tantamount to identifying functions with their graphs, i.e. with the relevant set of pairs $\langle$*argument, corresponding value*$\rangle$. And on this view, the codomain is irrelevant to the identity of a function. Then, in this example, f and g would count as the same function.
2. However, we noted $f : \mathbb{N} \to \mathbb{N}$ isn't a bijection, and $g : \mathbb{N} \to \mathbb{E}$ is. So if we want properties like being bijective to be intrinsic properties of a function, we will need to identify functions by their graphs (which, in the case of total functions, fixes their domain) *and* their co-domain.

The modern line is to prefer the second approach.

2. An inverse f^{-1} to a function f 'undoes' f's effect – in other words, applying f and then applying f^{-1} to the result takes us back to where we started. More carefully, if $f\colon A \to B$ is a function, then $f^{-1}\colon B \to A$ is an inverse to f iff for all $y \in B$, $f^{-1}(y) = x$ iff $f(x) = y$, which implies $f^{-1}(f(x)) = x$.
   (a) Prove that if f has an inverse, it has exactly one.
   (b) Prove that f has an inverse iff it is a bijection.
   (c) Assuming $f\colon A \to B$ has an inverse, what function is defined by $I(x) = f(f^{-1}(x))$?
   (d) Prove that $f\colon A \to B$ is a bijection, its inverse is a bijection.

(a) If g and h are both inverse to $f\colon A \to B$, then by definition, for all $y \in B$, $g(y) = x$ iff $f(x) = y$ iff $h(y) = x$, so $g(y) = h(y)$, so these inverses are indeed the *same* function.

(b) i. Suppose $f\colon A \to B$ is bijective. Then for every $y \in B$, there is one (because f is surjective) and only one (because f is injective) $x \in A$ such that $f(x) = y$, so we can indeed legitimately define $f^{-1}\colon B \to A$ by putting $f^{-1}(y) = x$ iff $f(x) = y$ and get a total, single-valued function.

ii. Suppose $f\colon A \to B$ has an inverse. Then there is a function $f^{-1}\colon B \to A$ such that $f^{-1}(y) = x$ iff $f(x) = y$. Since f^{-1} must take a value for every $y \in B$, for each $y \in B$ there must be a value of $x \in A$ such that $f(x) = y$ so y is surjective. And since for every $y \in B$, f^{-1} takes a unique value $x \in A$, if $f(x) = y$ and $f(x') = y$, then $x = x'$ so f is injective.

(c) Given we have

$$B \xrightarrow{f^{-1}} A \xrightarrow{f} B$$

the result of applying f^{-1} and then feeding the result to f gives us a function $I\colon B \to B$. And since $f^{-1}(y) = x$ iff $f(x) = y$, $f(f^{-1}(y)) = y$, so in fact I is the identity function which maps an object in B to itself.

(d) If $f\colon A \to B$ is a bijection, we know it has an inverse $f^{-1}\colon B \to A$.

Suppose $f^{-1}(x) = f^{-1}(y)$. Then $f(f^{-1}(x)) = f(f^{-1}(y))$, hence (by the previous argument) $x = y$, so f^{-1} is injective.

Now take any $y \in A$. There is an $x \in B$ such that $f(x) = y$ – for just put $x = f(y)$ and use the fact that $f^{-1}(f(y)) = y$. So f^{-1} is surjective too.

3. Given functions $f\colon A \to B$ and $g\colon B \to C$ (with the domain of g being the same as the codomain of f) their *composition* is the function $(g \circ f)$ (read 'g following f'), where $(g \circ f)\colon A \to C$, and for any $x \in A$, $(g \circ f)(x) = g(f(x))$.

   (a) When are $(h \circ j)$ and $(j \circ h)$ the same function?

   (b) When are $(h \circ (j \circ k))$ and $((h \circ j) \circ k)$ the same functions?

(a) Suppose we have $h\colon A \to B$ and $j\colon C \to D$. Then for there to be a composite function $(h \circ j)$, we need the domain of h to be the codomain of j, i.e. $A = D$. And for there to be a composite function $(j \circ h)$, we need the domain of j to be the codomain of h, i.e. $C = B$. So we have $h\colon A \to B$ and $j\colon B \to A$. So we then have $(j \circ h)\colon A \to A$ and $(h \circ j)\colon B \to B$. Hence finally, if these are to be same, we need $A = B$.

So $(h \circ j)$ and $(j \circ h)$ are the same function when, for some A, $h\colon A \to A$ and $j\colon A \to A$, and for all $x \in A$, $h(j(x)) = j(h(x))$.

(b) For $(h \circ (j \circ k))$ and $((h \circ j) \circ k)$ to be well-defined as functions, domains and codomains have to fit together so that, for some A, B, C, D, $k\colon A \to B$, $h\colon B \to C$ and $j\colon C \to D$. Then we have both $(h \circ (j \circ k))\colon A \to D$ and $((h \circ j) \circ k)\colon A \to D$. Moreover, for all $x \in A$, $(h \circ (j \circ k))(x) = h(j(k(x))) = ((h \circ j) \circ k)(x)$

In sum, assuming they exist, $(h \circ (j \circ k))$ and $((h \circ j) \circ k)$ are always the same function. Which is why we can henceforth drop bracketing without harm.

Now dropping unnecessary brackets, given functions $f\colon A \to B$ and $g\colon B \to C$ show that

(c) If f and g are injective, so is $g \circ f$.

(d) If f and g are surjective, so is $g \circ f$.

(e) If f and g are bijective, so is $g \circ f$.

Finally,

(f) If f and g are bijective, what is the inverse function to $g \circ f$?

(c) By the definition of 'injective', we have for all $x, y \in A$, $f(x) = f(y) \to x = y$, and for all $u, v \in B$, $g(u) = g(v) \to u = v$. Whence $g(f(x)) = g(f(y)) \to f(x) = f(y) \to x = y$. So, $g \circ f$ is injective.

(d) By the definition of 'surjective' applied to g, if we take an $z \in C$, there is some $y \in B$ such that $g(y) = z$. And by the definition of 'surjective' applied to f, for that $y \in B$ there will be an $x \in A$ such that $f(x) = y$. So taking that $z \in C$, there will be an $x \in A$ such that $g(f(x)) = z$. But z was an arbitrary member of C, so $g \circ f\colon A \to C$ is surjective.

(e) Trivial, given the conjunction of results (a) and (b).

(f) If f are g are bijective, so is $g \circ f$, so this composite function has an inverse. And $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$. For consider reversing the arrows in

$$A \xrightarrow{f} B \xrightarrow{g} C.$$

We get:

$$A \xleftarrow{f^{-1}} B \xleftarrow{g^{-1}} C.$$

with f^{-1} following g^{-1}.

4. Suppose the characteristic functions of the numerical properties P and Q are respectively c_P and c_Q. What are the characteristic functions of
   (a) The property n has when it isn't the case that Pn?
   (b) The property n has when either Pn and/or Qn?
   (c) The property n has when both Pn and Qn?
   (d) The property n has when, if Pn then Qn?

   Suppose a function $c : \mathbb{N} \to \{0, 1\}$ is given: must there always be a numerical property whose characteristic function is c?

Recall our definition of a characteristic function:

> The *characteristic function* of the numerical property P is the one-place function $c_P \colon \mathbb{N} \to \{0, 1\}$ such that if n is P, then $c_P(n) = 0$, and if n isn't P, then $c_P(n) = 1$.

And similarly for the characteristic functions of relations. It is perhaps more common to assign the values 0 and 1 the other way about (using 1 for as the truth-value 'true'). But we are here following Gödel himself, in 1934 paper 'On Undecidable Propositions of Formal Mathematical Systems' (p. 347 of Vol. 1 of the *Collected Works*):

> There shall correspond to each class or relation R a *representing function* ϕ such that $\phi(x_1, \ldots, x_n) = 0$ if $R(x_1, \ldots, x_n)$ and $\phi(x_1, \ldots, x_n) = 1$ if $\sim R(x_1, \ldots, x_n)$.

Simple calculations show:

(a) $c_{\sim P}(n) = 1 - c_P(n)$

(b) $c_{P \vee Q}(n) = c_P(n) \cdot c_Q(n)$

(c) $c_{P \wedge Q}(n) = 1 - (1 - c_P(n)) \cdot (1 - c_Q(n))$ [think De Morgan's Law!]

(d) $c_{P \to Q}(n) = (1 - c_P(n)) \cdot c_Q(n)$

Finally, suppose a function $c : \mathbb{N} \to \{0, 1\}$ is given. Then you could say: there is a corresponding property P which is had by n iff $c(n) = 0$: then trivially $c_P = c$.