Solutions: The language L_A

1. Given the conventions adopted in §5.1 and the definition of L_A given in §5.2, which of the following are (i) wffs of L_A, (ii) abbreviations of wffs of L_A, (iii) expressions of mathematical English?

   (a) $(1 + 2) = 3$.

   (b) $\forall x\, x + 0 = x$.

   (c) $\forall \mathsf{x}\, (\mathsf{x} + 0) = \mathsf{x}$.

   (d) $\forall \mathsf{x}(\mathsf{x} + 0 = \mathsf{x})$.

   (e) $\forall m \exists n\, n > m$.

   (f) $\forall \mathsf{m} \exists \mathsf{n}\, \mathsf{n} > \mathsf{m}$.

   (g) $\mathsf{x} + 1 \neq \mathsf{x}$.

   (h) $\xi + \mathsf{S0} \neq \xi$.

(a) (ii) An abbreviation of the L_A wff $(\mathsf{S0} + \mathsf{SS0}) = \mathsf{SSS0}$.

(b) (iii) A bit of informal arithmetic, borrowing quantifier notation ...

(c) (i) ... and its formal L_A counterpart.

(d) (i?) Note what we say at the very end of §5.1 concerning being casual about dropping brackets and inserting redundant ones for readability. So we can generously let this pass muster at least as an honorary wff of L_A. (There are two options: extending the official rules of L_A with rules for bracket-dropping and bracket-insertion to legitimize this, or treat this sort of thing as casual slang to be understood as replaced by a 'properly' bracketed wff. We needn't fuss too mucht about the difference.)

(e) (iii) Informal maths.

(f) We were generous about what count as variables of L_A, but we did indicate that these were to be from the beginning or end of the alphabet: but more importantly, '$>$' doesn't belong to the vocabulary L_A (though we could introduce it by definition).

(g) (ii) We are allowing open wffs, and we allow '$\neq$', so this counts as the abbreviation of the L_A wff $\neg \mathsf{x} + \mathsf{S0} = \mathsf{x}$.

(h) (iii?) L_A doesn't contain Greek symbols, so this isn't an L_A wff. We might (but in *IGT2* we don't) use such an expression in mathematicians' English in the context of talking *about* a gappy L_A expression, i.e. talk about the expression '$+\ \mathsf{S0}\ =$' with the Greek letters serving as place-markers for the gaps to be filled, the use of the same Greek letter in each slot indicating that the gaps are to be filled in the same way.

2. Give L_A wffs that express the following properties, relations, and functions (x, y, z range over natural numbers):

(a) The property of being an even number [give three alternatives].

(b) The property of being divisible by seven.

(c) The relation *being less than*.

(d) The property of being a composite number [i.e. not a prime].

(e) The function $x \mapsto x^2$.

(f) The relation of that x has to y when x is the square root of y.

(g) The relation that x has to y when x is a factor of y.

(h) The relation that x has to y and z when x is strictly between y and z.

(i) The function $x, y \mapsto |x - y|$.

(j) The property of being the sum of two primes.

Conclude that Goldbach's Conjecture can be stated in the language L_A.

(a) $\exists \mathsf{y}\, \mathsf{x} = \mathsf{y} + \mathsf{y}$; $\exists \mathsf{y}\, \mathsf{x} = \mathsf{y} \times \mathsf{SS0}$; $\neg\exists \mathsf{y}\, \mathsf{x} = \mathsf{S}(\mathsf{y} + \mathsf{y})$.

(b) $\exists \mathsf{y}\, \mathsf{x} = \mathsf{y} \times \mathsf{7}$.

(c) $\exists \mathsf{z}\, \mathsf{x} = \mathsf{y} + \mathsf{Sz}$. Abbreviate that $\mathsf{x} < \mathsf{y}$.

(d) $\exists \mathsf{y} \exists \mathsf{z}\, 1 < \mathsf{y} \wedge 1 < \mathsf{z} \wedge \mathsf{x} = \mathsf{y} \times \mathsf{z}$.

(e) $\mathsf{x} \times \mathsf{x} = \mathsf{y}$.

(f) $\mathsf{x} \times \mathsf{x} = \mathsf{y}$. [Note then that 'expressing in L_A' doesn't reflect the distinction between a functional relation R which holds between x and a unique y and the corresponding function $x \mapsto y$.]

(g) $\exists \mathsf{z}\, \mathsf{x} \times \mathsf{z} = \mathsf{y}$.

(h) $\exists \mathsf{u} \exists \mathsf{v}(\mathsf{y} + \mathsf{Su} = \mathsf{x} \wedge \mathsf{x} + \mathsf{Sv} = \mathsf{z})$.

(i) $(\exists \mathsf{u}(\mathsf{x} + \mathsf{u} = \mathsf{y}) \rightarrow \mathsf{x} + \mathsf{z} = \mathsf{y}) \wedge (\exists \mathsf{u}(\mathsf{y} + \mathsf{u} = \mathsf{u}) \rightarrow \mathsf{y} + \mathsf{z} = \mathsf{x})$.

(j) Use $\mathsf{Pr}(\mathsf{x})$ to abbreviate the wff (2) on p. 41 of *IGT2*, or to abbreviate the negation of the answer to (d) above. Then we want $\exists \mathsf{y} \exists \mathsf{z}(\mathsf{Pr}(\mathsf{y}) \wedge \mathsf{Pr}(\mathsf{z}) \wedge \mathsf{x} = \mathsf{y} + \mathsf{z})$. Abbreviate that $\mathsf{G}(\mathsf{x})$

Goldbach's conjecture is that any even number greater than 2 is the sum of two primes. So this will do: $\forall \mathsf{x}(\exists \mathsf{y}(\mathsf{x} = (2 \times \mathsf{y}) + 2) \rightarrow \mathsf{G}(\mathsf{x}))$.

3. The fundamental non-logical vocabulary of L_A is $\{\mathsf{S}, +, \times, \mathsf{0}\}$. Describe a variant language L'_A whose fundamental non-logical vocabulary is $\{+, \times, <, 0, 1\}$ (where the symbols have the obvious interpretations). Explain why it is a matter of indifference whether we choose to use the language L_A or L'_A.

The obvious thing to note is that wffs of L_A translate directly into wffs of L'_A by replacing every L_A-occurrence of $\mathsf{S}\tau$ for a term τ by $(\tau + 1)$.

Conversely, every wff of L'_A can be translated directly into a wff of L_A by replacing every L'_A-occurrence of 1 by $\mathsf{S0}$, and every occurrence of an expression $\sigma < \tau$ by an occurrence of $\exists \mathsf{v}\, \sigma + \mathsf{Sv} = \tau$ where v is the first variable not so far occurring in the wff we are in the course of stage-by-stage translating (so as to avoid clash-of-variables).

4. Consider the language L_A^β whose built-in non-logical vocabulary is $\{\mathsf{S}, +, \times, \mathsf{B}, \mathsf{0}\}$. Here B is a two-place function expression [written prefix], which expresses the function β, where $\beta(n, j)$ is the exponent of the prime p_j in the factorization of n (so is zero if p_j is not a factor of n).

   Give L_A^β wffs that express the numerical following properties, relations, and functions.

   (a) The property of being a power of 2.

   (b) The property of being prime.

   (c) The property c has when the exponent of p_2 in its prime factorization is twice the exponent of p_1, and the exponent of p_3 in its prime factorization is 2 times the exponent of p_1.

   (d) The relation c has to n when (i) the exponent of p_0 in the factorization of c is 1, and (2) for all $j < n$, the exponent of p_{j+1} in the factorization of c is $(j + 1)$ times the exponent of p_j.

   (e) The factorial function $x \mapsto x!$. [Hint: note that given a c which has that last relation to n, the exponent of p_n in its prime factorization must be $n!$.]

   (f) The exponential function $x, y \mapsto x^y$. [Use the same sort of trick you've just used to express the factorial.]

   What are the prospects for expressing the factorial and exponential functions in unaugmented L_A?

(a) $\forall \mathsf{y}\mathsf{B}(\mathsf{x}, \mathsf{Sy}) = \mathsf{0}$ [The exponent of every prime greater than 2 in the factorization of x is 0.]

(b) We could of course still use the wff $\mathsf{Pr(x)}$ which we defined before. Or we could put $\exists \mathsf{y} \forall \mathsf{z}(\mathsf{B}(\mathsf{x}, \mathsf{y}) = 1 \wedge (\mathsf{z} \neq \mathsf{y} \to \mathsf{B}(\mathsf{x}, \mathsf{z}) = \mathsf{0}))$.

(c) $\mathsf{B}(\mathsf{x}, 2) = 2 \times \mathsf{B}(\mathsf{x}, 1) \ \wedge \ \mathsf{B}(\mathsf{x}, 3) = 3 \times \mathsf{B}(\mathsf{x}, 2)$.

(d) $\mathsf{B}(\mathsf{x}, \mathsf{0}) = 1 \wedge \forall \mathsf{z}(\mathsf{z} < \mathsf{y} \to \mathsf{B}(\mathsf{x}, \mathsf{Sz}) = \mathsf{Sz} \times \mathsf{B}(\mathsf{x}, \mathsf{z}))$. Abbreviate that $\mathsf{F}(\mathsf{x}, \mathsf{y})$.

(e) $\exists \mathsf{c}(\mathsf{F}(\mathsf{c}, \mathsf{x}) \wedge \mathsf{B}(\mathsf{c}, \mathsf{x}) = \mathsf{y})$. Why does this work? Because it says there is a number c, which – taking its prime factors up to p_x – factorizes as $p_0^1 \cdot p_1^{1\times 1} \cdot p_2^{2\times 1\times 1} \cdot p_3^{3\times 2\times 1\times 1} \cdot \ldots \cdot p_x^{x!}$ and the exponent of p_x in the factorization is y. And this will evidently be true iff $y = x!$. [Though do think about both directions of that biconditional claim.]

(f) $\exists \mathsf{c}(\mathsf{B}(\mathsf{c}, \mathsf{0}) = 1 \wedge \forall \mathsf{u}(\mathsf{u} < \mathsf{y} \to \mathsf{B}(\mathsf{c}, \mathsf{Su}) = \mathsf{x} \times \mathsf{B}(\mathsf{c}, \mathsf{u})) \wedge \mathsf{B}(\mathsf{c}, \mathsf{y}) = \mathsf{z})$. That says there is a number c which – taking its prime factors up to p_y – factorizes as $p_0^1 \cdot p_1^{x\times 1} \cdot p_2^{x\times x\times 1} \cdot p_3^{x\times x\times x\times 1} \cdot \ldots \cdot p_y^{x^y}$ and the exponent of p_y in the factorization is z. Which will evidently be true iff $z = x^y$.

Here's an extra exercise for super-enthusiasts: Suppose we had started off instead with the language L_A^+ whose built-in non-logical vocabulary is $\{\mathsf{S}, +, \times, \mathsf{E}, \mathsf{0}\}$ where E expresses the two-place exponential function, so that the value of $\mathsf{E}(\overline{\mathsf{m}}, \overline{\mathsf{n}})$ is m^n. Show that in L_A^+ we can express the β-function, so that L_A^+ and L_A^β are equally expressive.

Can we use this sort of β-function trick in pure L_A and so define the factorial and the exponential there too?

What β in effect does is takes a 'code number' c and successive values of $\beta'(\overline{\mathsf{c}}, \mathsf{j})$ as we ratchet up values of j spit out a sequence of values. So it allows us to 'decode' codes for finite sequences of numbers. And this allows us in effect to talk about e.g. the sequence of numbers $0!, 1!, 2!, \ldots n!$ or the sequence of numbers $m^0, m^1, m^2, \ldots m^n$. Can we do this in L_A? It certainly isn't obvious that we can. But in Ch. 15 of *IGT2* we will find out how to do this trick, thanks to Gödel.