Appendix to the Study Guide

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Preface

The traditional menu for a first serious Mathematical Logic course covers basic first order logic with some model theory, the elementary theory of computability and related matters (like Gödel's incompleteness theorem), and introductory set theory. The *Beginning Mathematical Logic* Study Guide makes some recommendations for entry-level reading on these areas, topic-by-topic.

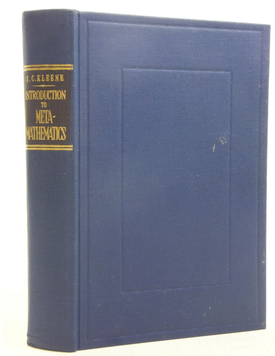
This Appendix complements the main Guide by adding book-by-book reviews of some of the more general texts that aim to cover an amount of this core menu. I give an indication of what they cover and – more importantly – of how they cover it, while commenting on style, accessibility, etc.

These wider-ranging books don't always provide the best currently available introductions to this or that particular area: but they can still be very useful for widening and deepening your understanding and they can reveal how topics from different areas fit together. I don't promise to (eventually!) discuss every worthwhile Big Book, or to give a similar level of coverage to those I do consider. But I'm working on the principle that even a somewhat patchy guide might still be useful. Undated entries were written about a decade ago. I am slowly revising them, and adding new entries.

The entry on a text starts with a general indication of its coverage, and some initial remarks. Then there is usually a more detailed description of contents with more specific comments. Finally, there's a summary verdict.

The books are listed in chronological order of first publication.

1 Kleene, 1952



First published seventy years ago, Stephen Cole Kleene's *Introduction to Metamathematics* (North-Holland, 1952; reprinted Ishi Press 2009: pp. 550) held the field for a while as a survey treatment of first-order logic (without going much past the completeness theorem) followed by a more in-depth treatment of the theory of computable functions and Gödel's incompleteness theorems.

In a later note about writing the book, Kleene remarks that up to 1985, about 17,500 copies of the English version of his text were sold, as were thousands of various translations (including a sold-out first print run of 8000 of the Russian translation). So this is a book with a quite pivotal influence on the education of whole generations of later logicians, and on their understanding of the fundamentals of recursive function theory and of the incompleteness theorems in particular.

And it isn't just nostalgia that makes old hands continue to recommend it. Kleene's book remains particularly lucid and accessible: it is often discursive, pausing to discuss the motivation behind formal ideas. It is still a pleasure to read – or at least, it ought to be a pleasure for anyone interested in logic enough to be reading this Appendix to the Guide! And, modulo relatively superficial presentational matters, you'll probably be struck by a sense of familiarity when reading through, as aspects of his discussions evidently shape many later textbooks (not least my own Gödel book).

The *Introduction to Metamathematics* remains a really impressive achievement: and not one to be admired only from afar, either.

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Some details Chs. 1–3 are introductory. There's a little about enumerability and countability (Cantor's Theorem); then a chapter on natural numbers, induction, and the axiomatic method; then a little tour of the paradoxes, and possible responses.

Chs. 4–7 are a gentle introduction to the propositional and predicate calculus and to a formal system which is in fact first-order Peano Arithmetic (you need to be aware that the identity rules are treated as part of the arithmetic, not the logic). Although Kleene's official system is Hilbert-style, he shows that 'natural deduction' introduction and elimination rules can be thought of as derived rules in his system, so it all quickly becomes quite user-friendly. (He doesn't at this point prove the completeness theorem for his predicate logic: as I said, things go quite gently at the outset!)

Ch. 8 starts work on 'Formal number theory', showing that his formal arithmetic has nice properties, and then defines what it is for a formal predicate to capture ('numeralwise represent') a numerical relation. Kleene then sets up Gödel coding and proves Gödel's incompleteness theorem, assuming a Lemma – eventually to be proved in his Chapter 10 – about the capturability of the relation 'm numbers a proof [in Kleene's system] of the sentence with code number n'.

Ch. 9 gives an extended treatment of primitive recursive functions, and then Ch. 10 deals with the arithmetization of syntax, yielding the Lemma needed for the incompleteness theorem.

Chs. 11–13 then give a nice treatment of general (total) recursive functions, partial recursive functions, and Turing computability. This is all very attractively done.

That leaves the final two chapters, in fact forming almost a quarter of the book under the heading 'Additional Topics'. In Ch. 14, after proving the completeness theorem for the predicate calculus without and then with identity, Kleene discusses the decision problem. And the final Ch. 15 discusses Gentzen systems, the normal form theorem, intuitionistic systems and Gentzen's consistency proof for arithmetic.

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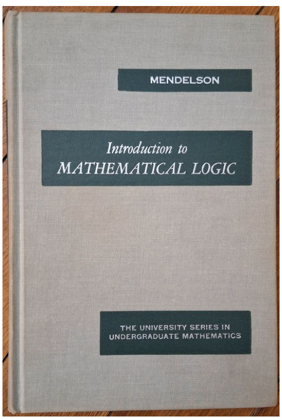
Summary verdict Seventy years on, Kleene's classic can *still* be warmly recommended as an enjoyable and illuminating presentation of this fundamental material, written by someone who was himself so closely engaged in the early developments back in the glory days. It should be entirely accessible if you have managed a modern introduction to FOL and perhaps already met Gödel's theorem, and it will certainly enrich and broaden your understanding.

Revised 2023

2 Mendelson, 1964, 2009



Elliot Mendelson's *Introduction to Mathematical Logic* (Van Nostrand, 1964: pp. 300) was first published in the distinguished and influential company of The University Series in Undergraduate Mathematics. It has been much used in graduate courses for philosophers since: a 4th edition was published by Chapman Hall in 1997 (pp. 440), with a slightly expanded 6th edition being published in 2015. I will here compare the first and fourth editions, as these are the ones I know best.

Even in the later editions this isn't, in fact, a very *big* Big Book (many of the added pages of the later editions are due to there now being answers to exercises). The length is kept under control in part by not covering a great deal, and in part by a certain brisk terseness. As the Series title suggests, the intended level of the book is upper undergraduate mathematics, and the book does broadly keep to that aim. Mendelson is indeed clear, taken slowly; however, his style is very much of its times, and will strike many modern readers as dry and rather old-fashioned. (Some of the choices of typography are not wonderfully pretty either, and this can make some pages look as if they will be harder going than they really turn out to be.)

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Some details After a brief introduction, Ch. 1 is on the propositional calculus. It covers semantics first (truth-tables, tautologies, adequate sets of connectives), then an axiomatic proof system. The treatments don't change much between editions, and will probably only be of interest if you've never encountered a Hilbert-style axiomatic system before.

The fine print of how Mendelson regards his symbolic apparatus is interesting: if you read him carefully, you'll see that the expressions in his formal systems are not sentences – not expressions of the kind that, on interpretation, can be true or false – but are *schemata*, what he calls statement forms. But this relatively idiosyncratic line about how the formalism is to be read, which for a while (due to Quine's influence) was oddly popular among philosophers, doesn't much affect the development.

Ch. 2 is on quantification theory, again in an axiomatic style. The fourth edition adds to the end of the chapter more sections on model theory: there is a longish section on ultra-powers and non-standard analysis, then there's (too brief) a nod to semantic trees, and finally a new discussion of quantification allowing empty domains. The extra sections in the fourth edition are a definite bonus: without them, there is nothing special to recommend this chapter, if you have worked through the suggestions on FOL in the Study Guide.

Ch. 3 is titled 'Formal number theory'. It presents a formal version of first-order Peano Arithmetic, and shows you can prove some expected arithmetic theorems within it. Then Mendelson defines the primitive recursive and the (total) recursive functions, shows that these are representable (capturable) in PA. It then considers the arithmetization of syntax, and proves Gödel's first incompleteness theorem and Rosser's improvement. The chapter then proves Church's Theorem about the decidability of arithmetic. One difference between editions is that the later proof of Gödel's theorem goes via the Diagonalization Lemma; another is that there is added a brief treatment of Löb's Theorem.

At the time of publication of the original addition, this Chapter was a quite exceptionally useful guide thorough the material. But now – at least if you've read an accessible later account like my *Gödel Without Tears* or the equivalent – then there is nothing especially to divert you here, except that Mendelson does go through *every* single stage of laboriously showing that the relation *m-numbers-a-PA-proof-of-the-sentence-numbered-n* is primitive recursive.

Ch. 4 is on set theory, and – rather unusually for a textbook – the system presented is NBG (von Neumann/Bernays/Gödel) rather than ZF(C). In the first edition, this chapter is under fifty pages, and evidently the coverage can't be very extensive and it also probably goes too rapidly for many readers. The revised edition doesn't change the basic treatment (much) but adds sections comparing NBG to a number of other set theories. So while this chapter certainly can't replace the introductions to set theory recommended in the Guide, it could be worth skimming briskly through the chapter in later editions to learn about NBG and other deviations from ZF.

The original Ch. 5 on effective computability starts with a discussion of Markov algorithms (again, unusual for a textbook), then treats Turing algorithms, then Herbrand-Gödel computability and proves the equivalence of the three approaches. There are discussions of recursive enumerability and of the Kleene-Mostowski hierarchy. And the chapter concludes with a short discussion of undecidable problems. In the later edition, the material is significantly rearranged, with Turing now taking pride of place and other treatments of computability relegated to near the end of the chapter; also more is added on decision problems. Since the introductory texts mentioned in the Guide don't talk about Markov or Herbrand-Gödel computability, you might well dip into the chapter briefly to round out your education!

I should mention the appendices. The first edition has a very interesting though brisk appendix giving a version of Schütte's variation on a Gentzen-style consistency proof for PA. Rather sadly, perhaps, this was missing from later editions though eventually restored in the sixth. The fourth edition adds an appendix on second-order logic.

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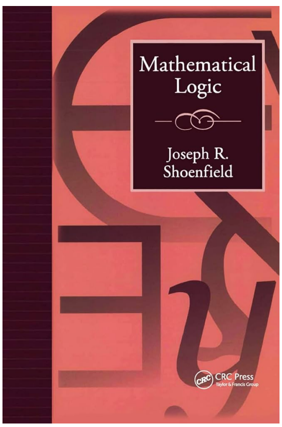
Summary verdict This book was a real achievement and extremely important in its time – the first modern mathematical logic textbook, educating generations of students. But there is now less reason to tackle this book end-to-end. It doesn't have the charm and readability of some other classics like Kleene 1952, and there are by now better separate introductions to each of the main topics.

You could skim the early chapters if you've never seen axiomatic systems of logic being used in earnest: it's probably good for the soul. The appendix that appears only in the first edition and sixth is interesting for enthusiasts. Look at the section on non-standard analysis in the revised editions. If set theory is your thing, you should dip and skim to get the headline news about NBG. And some might want to expand their knowledge of definitions of computation by looking at Ch. 5.

Revised 2023

3 Shoenfield, 1967



Joseph R. Shoenfield's *Mathematical Logic* (Addison-Wesley, 1967: pp. 334) was officially intended as 'a text for a first-year [maths] graduate course'. It has, over the years, been much recommended and much used (a lot of older logicians first learnt their more serious logic from it).

This book, however, is definitely hard going – a significant step or two up in level from Mendelson – though the added difficulty in mode of presentation seems to me not always to be necessary. I recall it as being daunting when I first encountered it as a student. Looking back at the book after a very long time, and with the benefit of greater knowledge, I have to say I am not any more enticed: it is still a tough read.

So this book can, I think, only be recommended to hard-core mathmos who already know a fair amount and can cherry-pick their way through the book. It does have heaps of hard exercises (without answers), and some interesting technical results are in fact buried there. But whatever the virtues of the book, they don't include approachability or elegance or particular student-friendliness.

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Some details Chs. 1–4 cover first order logic, including the completeness theorem. It has to be said that the logical system chosen is rebarbative. The primitives are $\neg$, $\vee$, $\exists$, and $=$. Leaving aside the identity axioms, the axioms are the instances of excluded middle, instances of $\varphi(\tau) \to \exists\xi\varphi(\xi)$, and then there are five rules of inference. So this neither has the cleanness of a Hilbert system nor the naturalness of a natural deduction system. Nothing is said to motivate this seemingly horrible choice as against others.

Ch. 5 is a brisk introduction to some model theory getting as far as the Ryll-Nardjewski theorem. I believe that the algebraic criteria for a first-order theory to admit elimination of quantifiers given here are original to Shoenfield. But this is surely all done very rapidly (unless you are using it as a terse revision course from quite an advanced base, going beyond what you will have picked up from the reading suggested in the Guide).

Chs. 6–8 cover the theory of recursive functions and formal arithmetic. The take-it-or-leave-it style of presentation continues. Shoenfield defines the recursive functions as those got from an initial class by composition and regular minimization: again, no real motivation for the choice of definition is given (and e.g. the definition of the primitive recursive functions is relegated to the exercises). Unusually for a treatment at this sort of level, the discussion of recursion theory in Ch. 8 goes far enough to cover a Gödelian 'Dialectica'-style proof of the consistency of arithmetic, though the presentation once more wins no prizes for accessibility.

Ch. 9 on set theory is perhaps the book's real original *raison d'être*; in fact, it is a quarter of the whole text. The discussion starts by briskly motivating the ZF axioms by appeal to the conception of the set universe as built in stages (an approach that has become very common but at the time of publication was I think much less usually articulated); but this isn't the place to look for an in depth development of that idea. For a start, there is Shoenfield's own article 'The axioms of set theory', *Handbook of mathematical logic*, ed. J. Barwise, (North-Holland, 1977) pp. 321–344.

We get a brusque development of the elements of set theory inside ZF (and then ZFC), and something about the constructible universe. Then there is the first extended textbook presentation of Cohen's 1963 independence results via forcing, published just four years previous to the publication of this book: set theory enthusiasts might want to look at this to help round out their understanding of the forcing idea. The discussion also touches on large cardinals.

This last chapter was in some respects a highly admirable achievement in its time: but it is equally surely not *now* the best place to start with set theory in general or forcing in particular, given the availability of later presentations.

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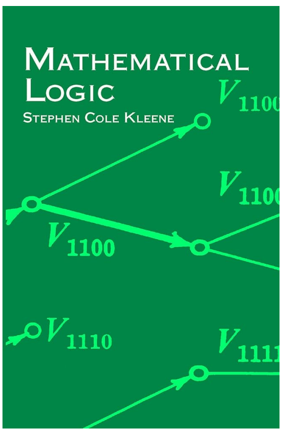
Summary verdict This is pretty tough going. Now surely only for *very* selective dipping into by already-well-informed enthusiasts.

4 Kleene, 1967



In the preface to his *Mathematical Logic* (John Wiley 1967, Dover reprint 2002: pp. 398), Stephen Cole Kleene writes "After the appearance in 1952 of my *Introduction to Metamathematics*, written for students at the first-year graduate level, I had no expectation of writing another text. But various occasions arose which required me to think about how to present parts of the same material more briefly, to a more general audience, or to students at an earlier educational level. These newer expositions were received well enough that I was persuaded to prepare the present book for undergraduate students in the Junior year."

You'd expect, therefore, that this later book would be more accessible, a friendlier read, than Kleene's remarkable *IM*. But in fact, this doesn't actually strike me as especially the case. In fact, I'd still recommend reading the older book, augmented by one chapter of this later 'Little Kleene'. To explain:

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Some details The book divides into two parts. The first part, 'Elementary Mathematical Logic' has three chapters. Ch. 1 is on the propositional calculus (including a Kalmár-style completeness proof). This presents a Hilbert-style proof system with an overlay of derived rules which look rather natural-deduction-like (but aren't the real deal) There is a lot of fussing over details in rather heavy-handed ways. I couldn't recommend anyone nowadays *starting* here, while if you've already read a decent treatment of the propositional calculus (and e.g. looked at Mendelson to see how things work in a Hilbert-style framework) you won't get much more out of this.

Much the same goes for the next two chapters. Ch. 2 gives an axiomatic version of the predicate calculus without identity, and Ch. 3 adds identity. (Note, a completeness proof doesn't come until the final chapter of the book). Again, these chapters are not done with a sufficiently light touch to make them a particularly attractive read now.

The second part of the book is titled 'Mathematical Logic and the Foundations of Mathematics'. Ch.4 is basically an abridged version of the opening three chapters of *IM*, covering the paradoxes, the idea of an axiomatic system, introducing formal number theory. You might like to read in particular §§36–37 on Hilbert vs. Brouwer and 'metamathematics'.

Ch. 5 is a sixty page chapter on 'Computability and Decidability'. Kleene is now on his home ground, and he presents the material (some original to him) in an attractive and illuminating way, criss-crossing over some of the same paths trodden in later chapters of *IM*. In particular, he uses arguments for incompleteness and undecidability turning on use of the Kleene T-predicate (compare §33.7 of *IGT1* or §43.8 of *IGT2*). This chapter is certainly worth exploring.

Finally, the long Ch. 6 proves the completeness theorem for predicate logic by Beth/Hintikka rather than by Henkin (as we would now think of it, he in effect shows the completeness of a tree system for logic in the natural way). But nicer versions of this approach are available. The last few sections cover some supplementary material (on Gentzen systems, Herbrand's Theorem, etc.) but again I think all of it is available more accessibly elsewhere

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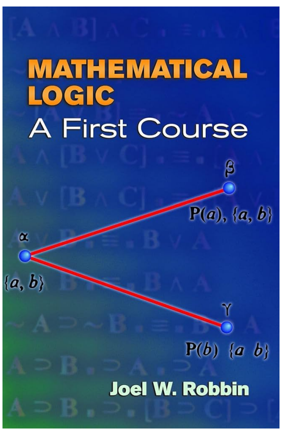
Summary verdict Do read Chapter 5 on computability, incompleteness, decidability and closely related topics. This is nicely done, complements Kleene's earlier treatment of the same material, and takes an approach which is interestingly different from what you will mostly see elsewhere.

5 Robbin, 1969



Joel W. Robbin's *Mathematical Logic: A First Course* (W. A Benjamin, 1969, Dover reprint 2006: pp. 212) is not exactly a 'Big Book'. The main text is just 170 pages long. But it does range over both formal logic (first-order and second-order), and formal arithmetic, primitive recursive functions, and Gödelian incompleteness.

Robbin, as you might guess, has to be pretty brisk (in part, he achieves brevity by leaving quite a few significant results to be proved as more or less challenging exercises). However, the book remains comparatively approachable and it has some nice and unusual features for which it can be recommended.

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Some details Ch. 1 is on the propositional calculus. Robbin presents an axiomatic system whose primitives are $\rightarrow$ and $\perp$ – or rather, in his notation, $\supset$ and $\mathfrak{f}$. The system, including the 'dotty' syntax which gives us wffs like $p_1 \supset p_2 \;.\!\supset\!.\; p_1 \supset \mathfrak{f}$, is a version of Alzono Church's system in his *Introduction to Mathmatical Logic*, Vol. 1 (1944/1956), except that where Church lays down three specific wffs as axioms and has a substitution rule for deriving variant wffs, Robbin lays down three axiom schemas. (I should really say something about Church's classic book in this Appendix: but that's for another day!)

As in later chapters, Robbin buries some interesting results in the extensive exercises. Here's one, pointed out to me by David Auerbach. Robbin defines negation in the obvious way from his two logical primitives, so that $\sim \varphi =_{\text{def}} (\varphi \rightarrow \mathfrak{f})$. And then his three axiom schemas can all be stated in terms of $\supset$ and $\sim$, and his one rule is modus ponens. This system is complete. However, if we take the alternative language with $\supset$ and $\sim$ *primitive*, then the same deductive system (with the same axioms and rules) is *not* complete. That's a nice little surprise, and it is worth trying to work out just why it is true.

Ch. 2 briefly covers first-order logic, including the completeness theorem. Then Ch. 3 introduces what Robbin calls 'First-order (Primitive) Recursive Arithmetic' (RA). Robbin defines the primitive recursive functions, and then defines a language which has a function expression for each p.r. function f (the idea is to have a complex function expression built up to reflect a full definition of f by primitive recursion and/or composition ultimately in terms of the initial functions). RA has axioms for the logic plus axioms governing the expressions for the initial functions, and then there are axioms for dealing with complex functional expressions in terms of their constituents. RA also has all instances of the induction schema for open wffs of the language (so – for cognoscenti – this is a stronger theory than what is usually called Primitive Recursive Arithmetic these days, which normally has induction only for quantifier-free wffs).

Ch. 4 explores the arithmetization of syntax of RA. Since RA has every p.r. function built in, we don't then have to go through the palaver of showing that we are dealing with a theory which can represent all p.r. functions (in the way we have to if we take standard PA as our base theory of interest). So in Ch. 5 Robbin can prove Gödel's incompleteness theorem for RA in a more pain-free way.

Ch. 6 then turns to second-order logic, introduces a version of second-order PA_2 with just the successor relation as primitive non-logical vocabulary. Robbin shows that all the p.r. functions can be explicitly defined in PA_2, so the incompleteness theorem carries over.

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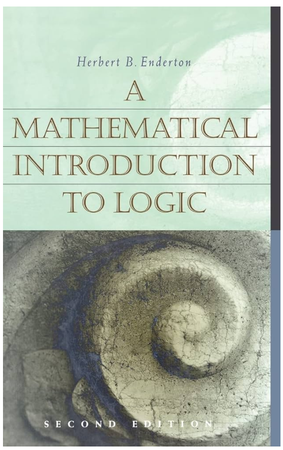
Summary verdict Robbin's book offers a different route through a rather different selection of material than is usual, accessibly written and still worth reading (you will be able to go though quite a bit of it pretty rapidly if you are already up to speed with the relevant basics). Look especially at Robbin's Ch. 3 for the unusually detailed story about how to build a language with a function expression for every p.r. function, and the last chapter for how in effect to do the same in PA_2.

6 Enderton, 1972, 2002



The first edition of Herbert B. Enderton's *A Mathematical Introduction to Logic* (Academic Press, 1972: pp. 295) rapidly established itself as much-used textbook among the mathematicians it was aimed towards. But it has also been used in math. logic courses offered to philosopers. A second edition was published in 2002, and a glance at the section headings indicates much the same overall structure: but there are many local changes and improvements, and I'll comment here on this later version of the book (which by now should be equally widely available in libraries).

Enderton's text deals with first order-logic and a smidgin of model theory, followed by a look at formal arithmetic, recursive functions and incompleteness. A final chapter covers second-order logic and some other matters.

A Mathematical Introduction to Logic eventually became part of a logical trilogy, with the publication of the wonderfully lucid *Elements of Set Theory* (1977) and *Computability Theory* (2010). The later two volumes strike me as masterpieces of exposition, providing splendid 'entry level' treatments of their material. The first volume, by contrast, is *not* the most approachable first pass through its material. It is good (often *very* good), but I'd say at a notch up in difficulty from what some might be looking for in an *introduction* to the serious study of first-order logic and/or incompleteness.

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Some details After a brisk Ch. 0 ('Some useful facts about sets', for future reference), Enderton starts with a 55 page Ch. 1, 'Sentential Logic'. Some might think this chapter to be slightly odd. For the usual motivation for separating off propositional logic and giving it an extended treatment at the beginning of a book at this level is that this enables us to introduce and contrast the key ideas of semantic entailment and of provability in a formal deductive system, and then explain strategies for soundness and completeness proofs, all in a helpfully simple and uncluttered initial framework. But (except for some indications in final exercises) there is no formal proof system mentioned in Enderton's chapter.

So what does happen in this chapter? Well, we do get a proof of the expressive completeness of $\{\wedge, \vee, \neg\}$, etc. We also get an exploration (which can be postponed) of the idea of proofs by induction and the Recursion Theorem, and based on these we get proper proofs of unique readability and the uniqueness of the extension of a valuation of atoms to a valuation of a set of sentences containing them (perhaps not the most inviting things for a beginner to be pausing long over). We get a direct proof of compactness. And we get a first look at the ideas of effectiveness and computability.

The core Ch. 2, 'First-Order Logic', is over a hundred pages long, and covers a good deal. It starts with an account of first-order languages, and then there is a lengthy treatment of the idea of truth in a structure. This is pretty clearly done and mathematicians should be able to cope quite well (but does Enderton forget his officially intended audience on p. 83 where he throws in an unexplained commutative diagram?). Still, readers might sometimes appreciate rather more explanation (for example, surely it would be worth saying a bit more than that 'In order to define 'σ is true in $\mathfrak{A}$' for sentences σ and structures $\mathfrak{A}$, we will find it desirable [sic] first to define a more general concept involving wffs', i.e. satisfaction by sequences).

Enderton then at last introduces a deductive proof system (110 pages into the book). He chooses a Hilbert-style presentation, and if you are not already used to such a system, you won't get much of a feel for how they work, as there are very few examples before the discussion turns to metatheory (even Mendelson's presentation of a similar Hilbert system is here more helpful). Then, as you'd expect, we get the soundness and completeness theorems. The proof of the latter by Henkin's method *is* nicely chunked up into clearly marked stages, and again a serious mathematics student should cope well: but this is still not, I think, a 'best buy' among initial presentations.

The chapter ends with a little model theory – compactness, the LS theorems, interpretations between theorems – all rather briskly done, and there is an application to the construction of infinitesimals in non-standard analysis which is surely going to be *too* compressed for a first encounter with the ideas.

Ch. 3, 'Undecidability', is also a hundred pages long and again covers a great deal. After a preview introducing three somewhat different routes to (versions of) Gödel's incompleteness theorem, we initially meet:

1. A theory of natural numbers with just the successor function built in (which is shown to be complete and decidable, and a decision procedure by elimination of quantifiers is given).

2. A theory with successor and the order relation (also shown to admit elimination of quantifiers and to be complete).

3. Presburger arithmetic (shown to be decidable by a quantifier elimination procedure, and shown not to define multiplication)

4. Robinson Arithmetic with exponentiation.

The discussion then turns to the notions of definability and representability. We are taken through a long catalogue of functions and relations representable in Robinson-Arithmetic-with-exponentiation, including functions for encoding and decoding sequences. Next up, we get the arithmetization of syntax done at length, leading as you'd expect to the incompleteness and undecidability results.

But we aren't done with this chapter yet. We get (sub)sections on recursive enumerability, the arithmetic hierarchy, partial recursive functions, register machines, the second incompleteness theorem for Peano Arithmetic, applications to set theory, and finally we learn how to use the β-function trick so we can get take our results to apply to any nicely axoimatized theory containing plain Robinson Arithmetic.

As is revealed by that quick description there really is a *lot* in Ch. 3. To be sure, the material here is not mathematically difficult in itself (indeed it is one of the delights of this area that the initial Big Results come so quickly). However, I do doubt that such an action-packed presentation is the best way to first meet this material. It would, however, make for splendid revision-consolidation-extension reading after tackling e.g. my Gödel book.

The final Ch. 4 is much shorter, on 'Second-Order Logic'. This goes *very* briskly at the outset. It again wouldn't be my recommended introduction for this material, though it could make useful supplementary reading for those wanting to get clear about the relation between second-order logic, Henkin semantics, and many-sorted first-order logic.

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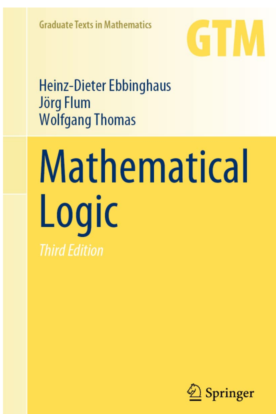
Summary verdict To repeat, *A Mathematical Introduction to Logic* is good in many ways, but is – in my view – often a step or two more difficult in mode of presentation than will suit many readers wanting an introduction to the material it covers. However, if you have already read an entry-level presentation of first order logic (e.g. Chiswell/Hodges) then you could read Chs 1 and 2 as revision/consolidation. And if you have already read an entry-level presentation on incompleteness (e.g. my book) then it could be very well worth reading Ch. 3 as bringing the material together in a somewhat different way.

7 Ebbinghaus, Flum & Thomas, 1978, 2021



Mathematical Logic by H.-D. Ebbinghaus, J. Flum and W. Thomas (Springer, 3rd edition 2021, pp. 304) was first published in German in 1978. A second edition appeared in English in 1994, in a series 'Undergraduate Texts in Mathematics'. The latest 2021 edition, adding just a little new material, is now in Springer's series 'Graduate Texts in Mathematics', which I think better reflects the level of quite a bit of the book.

EFT's book is often praised and is I believe quite widely used. But revisiting the latest edition, I can't find myself wanting to recommend it as a particularly good place to start, whether for philosophers or for mathematicians. The presentation of the core material on the syntax and semantics of first-order logic in the first half of the book is done more elegantly elsewhere. In the second half of the book, the chapters do range widely across interesting material. But again most of the discussions will probably go too quickly if you haven't encountered the topics before, and – if you want revision/amplification of what you already know – you will mostly do better elsewhere. I'll pick out the later chapters which can be most recommended.

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Some details about Part A The book is divided into two parts. EFT start Part A with a gentle opening chapter talking about a couple of informal mathematical theories (group theory, the theory of equivalence relations), giving a couple of simple informal proofs in those theories. They then stand back to think about what goes on in the proofs (where an arbitrary item in a domain is selected, a result proved and then universally generalized); and they introduce the project of formalization. So far, so nice.

Ch. 2 describes the syntax of first-order languages, and proves some unique parsing results relatively painlessly. (EFT take the usual line of using the same symbols for both free and bound variables which causes the usual extra work. Also, a minor annoyance, they use '≡' rather than '=' as the object language sign for identity.)

Ch. 3 does the semantics for FOL. The presentation goes at a fairly gentle pace, with some useful asides (e.g. on handling the many-sorted languages of informal mathematics using a many-sorted calculus vs. use restricted quantifiers in a single-sorted calculus). EFT though do make quite heavy work of some points of detail, but overall this is an approachable version of a standard story.

Ch. 4 is called 'A Sequent Calculus'. And here I am less happy.

For a start (albeit a minor point, but one that badly affects readability), instead of writing a sequent as '$\Gamma \vdash \varphi$', or '$\Gamma \Rightarrow \varphi$', or even '$\Gamma\colon \varphi$', EFT just write an unpunctuated '$\Gamma\ \varphi$'. They even write the unpunctuated 'Γ, φ, ψ' for '$\Gamma, \varphi, \vdash \psi$'. Strange!

EFT have by this point decided to take only $\neg$, $\vee$, and $\exists$ as basic, and give rules just for these. Given the paucity of basic operators, EFT are not aiming for natural deduction in sequent form; nor are they aiming for a classical system which nicely relates to an intuitionistic subsystem. Proofs are simple linear sequences of sequents (no Gentzen-style trees). The resulting system is economical, and we quickly e.g. get a cut rule for free. But the distinction between structural and operation rules usually highlighted by presentations of a sequent calculus is arguably somewhat glossed over. So I'm not sure that I'd recommend this as the first-proof system to encounter.

Ch. 5 gives a Henkin completeness proof for first-order logic. For my money, there's too much symbol-bashing and not enough motivating chat here. And I don't think it is good exegetical policy to complicate matters as EFT do by going straight for a proof for the predicate calculus with identity, though they are not alone in this.

Ch. 6 is rather briskly about The Löwenheim-Skolem Theorem, compactness, and elementarily equivalent structures (clear enough, and would be good revision material).

Ch. 7, 'The Scope of First-Order Logic' is really rather odd. It briskly argues that first-order logic is the logic for mathematics (readers of Shapiro's book on second-order logic won't be so quickly convinced!). The reason given is that we can reconstruct (nearly?) all mathematics in first-order ZF set theory – which the authors then proceed to give the axioms for. These few pages surely wouldn't help if you have never seen the axioms before and don't already know about the project of doing-maths-inside-set-theory.

Finally in Part A, there's rather ill-written chapter on normal forms, on extending theories by definitions, and (badly explained) on what the authors call 'syntactic interpretations'.

◊

Some details about Part B The second part of the book discusses a number of rather scattered topics. It kicks off with a rather nice little chapter on extensions of first-order logic, more specifically on second-order logic, on $\mathcal{L}_{\omega_1\omega}$ (which allows infinitely long conjunctions and disjunctions), and $\mathcal{L}_Q$ (logic with quantifier Qx, 'there are uncountably many x such that ... '). Modest ambition, clearly done.

Then Ch. 10 is on 'Limitations of the Formal Method', and is decidedly over-ambitious. In 35 pages, EFT aim to talk about register machines, the halting problem for such machines, the undecidability of first-order logic, Trakhtenbrot's theorem and the incompleteness of second-order logic, Gödel's incompleteness theorems, and more. This would surely just be too rushed if you'd not seen this material before. And – while this is pretty clear – if you have seen these themes explored before there are better sources for revising/consolidating/extending your knowledge. The latest edition then adds a new section on the decidability of Presberger Arithmetic and a substantial (but rather out of place?) section on the decidability of a form of successor arithmetic.

Ch. 11 is another hefty fifty pages, on 'Free Models and Logic Programming'. This starts with Herbrand's Theorem and some related matters. And then – at last – we meet some basics of propositional logic that we haven't met before, and get introduced to propositional resolution. Then we get more first-order resolution and logic programme. I can't say I found these discussions attractive.

Ch. 12 is back to core model theory, Fraïssé's Theorem and Ehrenfeucht Games: but you'll again find better treatments elsewhere, this time in books dedicated to model theory.

Finally, there is an interesting (though quite tough) concluding chapter on Lindström's Theorems which show that there is a sense in which standard first-order logic occupies a unique place among logical theories.

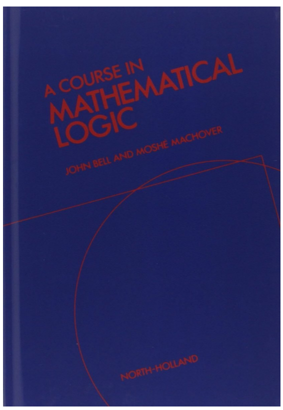
◊

Summary verdict This is a perfectly respectable book, but the core material in Part A of the book is covered better (more accessibly, more elegantly) elsewhere.

Of the supplementary chapters in Part B, the two chapters that stand out as worth looking at are perhaps Ch. 9 on extensions of first-order logic, and Ch. 13 (though not easy) on Lindström's Theorems.

Revised 2023

8 Bell & Machover, 1977



Once upon a long time ago, John Bell and Moshé Machover ran a notable one-year masters-level programme on mathematical logic and the foundations of mathematics at London University. They then developed their lecture notes into *A Course in Mathematical Logic* published by North-Holland in 1977. This is a very substantial book of 599 pages, rightly very well-regarded back in the day. But, almost fifty years on, the question inevitably arises: what does it have to offer the reader today compared with more recent texts (in particular, how well is it likely to work for a student launching into a course of self-study?).

As the authors note, various parts of the Course had their origins in different lecture courses, and they can often be tackled independently of each other. For example, you don't need to know the particular details of Bell and Machover's account of FOL in order to cope with their later discussion of formal arithmetics. It is therefore quite reasonable to chunk up the book into discrete parts, and to assess these separately. So that's what I'll do.

◊

Chs 1–3, *Propositional and first-order logic* (pp. 124). There is a lot to like about the first chapter here – in particular the way that both tableaux and axiomatic systems are introduced and interrelated. (Unsigned) tableaux are nicely motivated immediately after the truth-functional semantics for the basic connectives is defined (and we get a proof too that a version of excluded middle could be conservatively added as a tableaux rule). Soundness and a (weak) completeness theorem for tableaux are snappily proved. Then we meet a Hilbert-style proof system, and it is shown directly (i.e. by syntactic proof-manipulations) that this warrants the same deductions as the tableaux system, establishing that the new proof system is complete too. Then we get a proof of weak completeness for our Hilbert system by Kalmar's method, and a proof of strong completeness by Zorn's Lemma and the construction of maximal consistent sets. This could be very useful reading for many, drawing various ideas together and showing how they interrelate.

But I can't in the same way recommend the similarly structured chapters on FOL. Although they start well enough, the story about FOL – in particular, the laboured discussion of substitution – soon becomes rather too heavy-handed, and the version of a tableau system for FOL is far from the nicest.

◊

Ch. 4, *Boolean algebras* (pp. 36) This is a nice stand-alone chapter, and it still makes a recommendable introductory account – especially §§1–5.

Ch. 5, *Model theory* (pp. 65) Somewhat action-packed, and lacking some of the helpful classroom asides we find in earlier chapters. There are now more accessible treatments of elementary model theory.

Chs 6–8, *Recursion theory and arithmetic* (pp. 174) Once upon a time, this group of chapters would have been particularly interesting in virtue of its early account of the then relatively-recent MRDP theorem and the use of that theorem in proving further key results. Still pretty readable, but this is a topic area with some wonderful alternative texts. Though for one alternative option on recursion theory and arithmetic, we should certainly note Machover's more approachable reworking of some of the same material in his own later, shorter, book (though he doesn't actually prove the MRDP theorem there, referring back to the details here).

Ch 9, *Intuitionistic first-order logic* (pp. 59). There is a useful initial motivating discussion in §§1-4. But we don't get the clearest of accounts of how the Beth/Fitting tableaux system which is introduced next is supposed to work: de Swart in his chapter on intuitionism, for example, does better. And the rest of the chapter doesn't give e.g. the nicest introduction to Kripke semantics either. So not the place to start.

Ch. 10, *Set theory* (pp. 72). After more routine introductory sections, the remaining sections – including §5 on reflection principles, §7 on absoluteness, §8 on constructible sets, §9 on the consistency of AC and GCH – could well still be useful supplementary reading for those who already know some elementary set theory. But of course there are many alternatives!

Ch. 11, Non-standard analysis (pp. 45). This chapter is a sophisticated but somewhat opaque treatment, a level or two up from most of what has gone before, and rather too remote from the accessible but intriguing entry-level considerations we usually meet in introductory accounts of Robinson-style constructions. Certainly not for the faint-hearted.

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A very brisk summary verdict An exceptional book in its day. Still worth revisiting in part.
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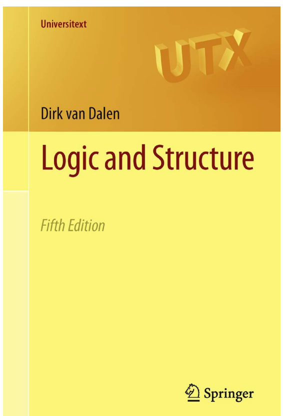
9 Van Dalen, 1980, 2012



Dirk van Dalen's popular *Logic and Structure* (Springer: 263 pp. in the most recent edition) was first published in 1980, and has now gone through a number of editions. It is widely used and has a lot to recommend it. A very substantial chapter on incompleteness was added in the fourth edition in 2004. A fifth edition published in 2012 adds a further new section on ultraproducts. Comments here apply to these last two editions.

$\Diamond$

Some details Ch. 1 on 'Propositional Logic' gives a presentation of the usual truth-functional semantics, and then a natural deduction system (initially with primitive connectives $\rightarrow$ and $\bot$). This is overall pretty clearly done – though really rather oddly, although van Dalen uses in his illustrative examples of deductions the usual practice of labelling a discharged premiss with numbers and using a matching label to mark the inference move at which that premiss is discharged, he doesn't pause to explain the practice in the way you would expect. Van Dalen then gives a standard Henkin proof of completeness for this cut-down system, before re-introducing the other connectives into his natural deduction system in the last section of the chapter. Compared with Chiswell and Hodges, this has a somewhat less friendly, more conventional, mathematical look-and-feel: but this is still an accessible treatment, and will certainly be very readily manageable if you've read C&H first. (It should be noted that van Dalen can be surprisingly slapdash. For example, a tautology is defined is defined to be an (object-language) proposition which is always true. But then the meta-linguistic schema $\varphi \wedge \psi \leftrightarrow \psi \wedge \varphi$ is said to be a tautology. He means the instances are tautologies; compare Mendelson who really *does* think tautologies are schemata. Again, van Dalen presents his natural deduction system, says he is going to give some 'concrete cases', but then presents not arguments in the object language, but schematic templates for arguments, written out using 'φ's and 'ψ's again.)

Ch. 2 describes the syntax of a first-order language, gives a semantic story, and then presents a natural deduction system, first adding quantifier rules, then adding identity rules. Overall, this is pretty clearly done (though van Dalen reuses variables as parameters, which isn't the nicest way of setting things up). The approach to the semantics is to consider an extension of a first order language L with domain A to an augmented language L^A which has a constant $\overline{a}$ for every element $a \in A$; and then we can say $\forall x\varphi(x)$ is true if all $\varphi(\overline{a})$ are true. Fine: though it would have been good if van Dalen had paused to say a little more about the pros and cons of doing things this way rather than the more common Tarskian way that students will encounter. (Let's complain some more about van Dalen's slapdash ways. For example, he talks about '*the* language of a similarity type' in §2.3, but gives examples of different languages of the same similarity type in §2.7. He fusses unclearly about different uses of the identity sign in §2.3, before going on to make use of the symbolism ':=' in a way that isn't explained, and is different from the use made of it in the previous chapter. This sort of thing could upset the more pernickety reader.)

Ch. 3, 'Completeness and Applications', gives a pretty clear presentation of a Henkin-style completeness proof, and then the compactness and L-S theorems. The substantial third section on model theory goes rather more speedily, and you'll need some mathematical background to follow some of the illustrative examples. The final section newly added in the fifth edition on the ultraproduct construction speeds up again and is probably too quick to be useful to many.

Ch. 4 is quite short, on second-order logic. If you have already seen a presentation of the basic ideas, this quick presentation of the formalities could be helpful.

Ch. 5 is on intuitionism – this is a particular interest of van Dalen's, and his account of the BHK interpretation as motivating intuitionistic deduction rules, his initial exploration of the resulting logic, and his discussion of the Kripke semantics are quite nicely done (though again, this chapter will probably work better if you have already seen the main ideas in a more informal presentation before).

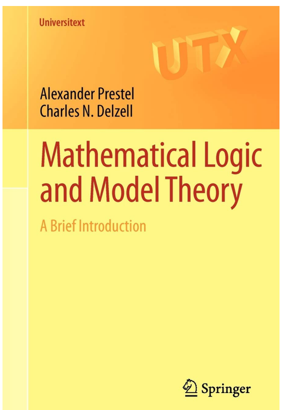
Ch. 6 is on proof theory, and in particular on the idea that natural deduction proofs (both classical and intuitionistic) can be normalized. Most readers will find a more expansive and leisurely treatment much to be preferred.

The final 50 page Ch. 7 *is* more leisurely. It starts by introducing the ideas of primitive recursive and partial recursive functions, and the idea of recursively enumerable sets, leading up to a proof that there exist effectively inseparable r.e. sets. We then turn to formal arithmetic, and prove that recursive functions are representable (because his version of PA does have the exponential built in, van Dalen doesn't need to tangle with the β-function trick.). Next we get the arithmetization of syntax and proofs that the numerical counterparts of some key syntactic properties and relations are primitive recursive. Then, as you would expect, we get the diagonalization lemma, and that is used to prove Gödel's first incompleteness theorem. We then get another proof relying on the earlier result that there are effectively inseparable r.e. sets, and going via the undecidability of arithmetic. The chapter finishes by announcing that there is a finitely axiomatized arithmetic strong enough to represent all recursive properties/relations, so the undecidability of arithmetic implies the undecidability of first-order logic. There's nothing, however, about the second theorem: so most students who get this far will want that bit more more. However, this chapter is all done pretty clearly, could probably be managed by good students as a first introduction to its topics, and would be very good revision/consolidatory reading for those who've already encountered this material.

◊

Summary verdict Revisiting this book, I find it a rather patchily uneven read. Although intended for beginners in mathematical logic, the level of difficulty of the discussions rather varies, and the amount of more relaxed motivational commentary also varies. As noted there are occasional lapses where van Dalen's exposition isn't as tight as it could be. So this is probably best treated as a book to be read *after* you've had a first exposure to the material in the various chapters: but then it should indeed prove pretty helpful for consolidating/expanding your initial understanding and then pressing on a few steps.

10 Prestel & Delzell, 1986, 2011

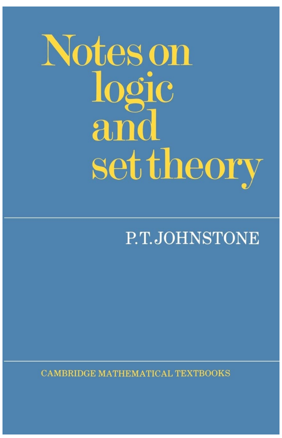


Alexander Prestel and Charles N. Delzell's *Mathematical Logic and Model Theory: A Brief Introduction* (Springer, 1986; English version 2011: pp. 193) is advertised as offering a 'streamlined yet easy-to-read introduction to mathematical logic and basic model theory'. Easy-to-read, perhaps, for those with a fair amount of mathematical background in algebra, for – as the Preface makes clear – the aim of the book is make available to interested mathematicians 'the best known model theoretic results in algebra'. The last part of the book develops a complete proof of Ax and Kochen's work on Artin's conjecture about Diophantine properties of p-adic number fields.

So this book is not really aimed at a likely reader of the *Beginning Mathematical Logic* Guide. Still, Ch. 1 is a crisp and clean 60 page introduction to first-order logic, that could be used as brisk and helpful revision material. And Ch. 2, 'Model constructions' gives a nice if pacey introduction to some basic model theoretic notions: again – at least for readers of this Guide – it could serve well to consolidate and somewhat extend ideas if you have already encountered at least some of this material before.

The remaining two chapters, 'Properties of Model Classes' and 'Model Theory of Several Algebraic Theories' are tougher going, and belong with the more advanced reading like Marker's book. But, for those who want to work through this material, it does strike me as well presented.

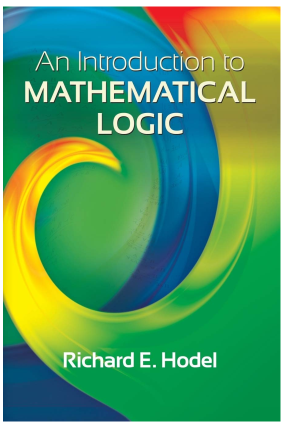
11 Johnstone, 1987



Peter T. Johnstone's *Notes on Logic and Set Theory* (CUP, 1987: pp. 111) is very short in page length, but very big in ambition. There is an introductory chapter on universal algebra, followed by chapters on propositional and first-order logic. Then there is a chapter on recursive functions (showing that a function is register computable if and only if computable, and that such functions are representable in PA). That is followed by four chapters on set theory (introducing the axioms of ZF, ordinals, AC, and cardinal arithmetic). And there is a final chapter 'Consistency and independence' on Gödelian incompleteness and independence results in set theory.

This is a quite remarkably action-packed menu for such a short book. True, the story is filled out a bit by the substantive exercises, but still surely this isn't the book to use for a first encounter with these ideas (even though it started life as notes for undergraduate lectures for the maths tripos). However, I would warmly recommend the book for revision/consolidation: its very brevity means that the Big Ideas get highlighted in a particularly uncluttered way, and particularly snappy proofs are given.

12 Hodel, 1995



Richard E. Hodel's *An Introduction to Mathematical Logic* (PWS Publishing, 1995, reprinted Dover Publications, 2013: pp. 491) was originally launched into the world by a relatively obscure publisher, but has now been taken up and cheaply republished by Dover. I hadn't heard of the book before a few people recommended it, commenting on the lack of any mention in earlier versions of this Guide.

The book covers first-order logic and some recursion theory, with a less usual – though not unique – feature being a full textbook treatment of Hilbert's Tenth Problem (the one about whether there is an algorithm which tells us when a polynomial equation $p(x) = 0$ has a solution in the integers).

So how does Hodel fare against his competitors? Overall, the book *is* pretty clearly written, though it does have a somewhat old-fashioned feel to it (you wouldn't guess, for example, that it was written some twenty years after George Boolos and Richard Jeffrey's *Computability and Logic*). And Hodel does give impressively generous sets of exercises throughout the book – including some getting the student to prove significant results by guided stages. However, I can't really recommend his treatment of logic in the first half of the book.

◊

Some details Ch. 1 'Background' is an unusually wide-ranging introduction to ideas the budding logician should get her head round early. We get a first informal pass at the notions of a formal system and of an axiomatized system in particular, the idea of proof by induction, a few notions about sets, functions and relations, the idea of countability, the ideas of an algorithmically computable function and of effective decidability. We even get a first pass at the idea of a recursive function (and a look at Church's Thesis about how the informal idea of being computable relates to the idea of recursiveness). This is very lucidly done, and can be recommended.

Ch. 2 is on 'The language and semantics of propositional logic' and is again pretty clearly done.

Ch. 3 turns to formal deductive systems for propositional logic. Unfortunately, Hodel chooses to work primarily with Shoenfield's system (the primitive connectives are $\neg$ and $\vee$, every instance of $\neg A \vee A$ is an axiom, and there are four rules of inference). I *really* can't see the attraction of this system among all the competitors, or why it should be thought as especially appropriate as a starting point for beginners. It neither has the naturalness of a natural deduction system, nor the austere Bauhaus lines of one of the more usual Frege-Hilbert axiomatic systems. The chapter does also consider other systems of propositional logic in the concluding two sections, but goes too quickly to be very helpful. So this key chapter is not a success, it seems to me.

Ch. 4 on 'First-order languages', including Tarskian semantics, is again pretty clear (and could be helpful to a beginner who is first encountering the ideas and is looking for reading to augment another textbook). But as we'd expect given what's gone before, when we turn to Ch. 5 on 'First-order logic' we can a continuance of the discussion of a Shoenfield-style system (except that Hodel takes $\forall$ rather than $\exists$ as primitive). Which is, by my lights, much to be regretted. The ensuing discussion of completeness, while perhaps a little laboured, is carefully structured with a good amount of signposting. But there are better presentations of first-order logic overall.

Ch. 6 is called 'Mathematics and logic' and touches on an assortment of topics about first-order theories and their limitations (and a probably rather-too-hasty look at set theory as an example).

The next two chapters form quite a nice unit. Ch. 7 discusses 'Incompleteness, undecidability and indefinability'. Recursive functions are defined Shoenfield-style as those arising from a certain class of initial functions by composition and regular minimization, which eases the proof that all recursive functions are representable (though doesn't do much to make recursiveness seem a natural idea to beginners). Then, by making the informal assumption that certain intuitively decidable relations are recursive, Hodel proves Gödel's incompleteness theorem, Church's Theorem and Tarski's Theorem. The next chapter fills in enough detail about recursive functions and relations to show how to lift that informal assumption. This seems all pretty clearly done, even if not a first-choice for real beginners.

Then Ch. 9 extends the treatment of computability by showing that the functions computable by an unlimited register machine are just the recursive ones: but, again this sort of thing is done at least as well in other books.

Finally, Ch. 10 deals with Hilbert's tenth problem (so the first five sections of Ch. 8 and then this chapter form a nice, stand-alone treatment of the negative solution of Hilbert's problem).

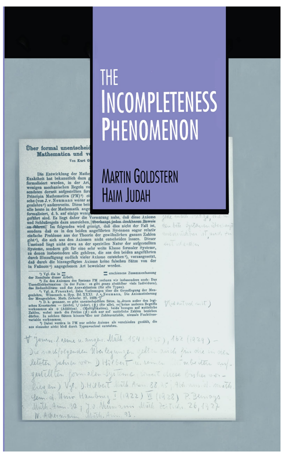
◊

Summary verdict Beginners could all usefully read Hodel's opening chapter, which is better than usual in setting the scene, and supplying the student with a useful toolkit of preliminary ideas. The presentation of first-order logic in Chs.

2-6 is based around an unattractive formal system, and while the discussion of the usual meta-theoretic results is pretty clear, it doesn't stand out from the good alternatives: so overall, I wouldn't recommend this as your first encounter with serious logic.

Students, however, might find Chs. 7 and 8 provide a nice complement to other discussions of Gödelian incompleteness and Church's and Tarski's Theorems. While more advanced students could revise their grip on basic definitions and results by (re)reading §§8.1–8.5 and then enjoy tackling Ch. 10 on Hilbert's Tenth Problem.

13 Goldstern & Judah, 1995



Half of Martin Goldstern and Haim Judah's *The Incompleteness Phenomenon: A New Course in Mathematical Logic* (A.K. Peters, 1995: pp. 247) is a treatment of first-order logic.

The rest of the book is two long chapters (as it happens, of just the same length), one on model theory, one mostly on incompleteness and with a little on recursive functions. So the emphasis on incompleteness in the title is somewhat misleading: the book is at least equally an introduction to some model theory. I have had this book recommended to me more than once, but I seem to be immune to its supposed charms (I too often don't particularly like the way that it handles the technicalities): your mileage may vary.

◊

Some details Ch. 1 starts by talking about inductive proofs in general, then gives a semantic account of sentential and then first-order logic, then offers a Hilbert-style axiomatic proof system.

Very early on, the authors introduce the notion of $\mathcal{M}$-terms and $\mathcal{M}$-formulae. An $\mathcal{M}$-term (where $\mathcal{M}$ is model for a given first-order language $\mathcal{L}$) is built up from $\mathcal{L}$-constants, $\mathcal{L}$-variables *and/or elements of the domain of* $\mathcal{M}$, using $\mathcal{L}$-function-expressions; an $\mathcal{M}$-formula is built up from $\mathcal{M}$-terms in the predictable way. Any half-awake student is initially going to balk at this. Re-reading the set-theoretic definitions of expressions as tuples, she will then realize that the apparently unholy mix of bits of language and bits of some mathematical domain in an $\mathcal{M}$-term is not actually incoherent. But she will rightly wonder what on earth is going on and *why*: our authors don't pause to explain why we might want to do things like this at the very outset. (A good student who knows other presentations of the basics of first-order semantics should be able to work out after the event what is going on in the apparent trickery of Goldstern and Judah's sort of story: but I really can't recommend *starting* like this, without a good and expansive explanation of the point of the procedure.)

Ch. 2 gives a Henkin completeness proof for the first-order deductive system given in Ch. 1. This has nothing special to recommend it, as far as I can see: there are many more helpful expositions available. The final section of the chapter is on non-standard models of arithmetic: Boolos and Jeffrey (Ch. 17 in their third edition) do this more approachably.

Ch.3 is on model theory. There are three main sections, 'Elementary substructures and chains', 'ultra products and compactness', and 'Types and countable models'. So this chapter – less than sixty pages – aims quite high to be talking e.g. about ultraproducts and about omitting types. You could indeed usefully read it after working through e.g. Manzano's book: but I certainly don't think this chapter makes for an accessible and illuminating first introduction to serious model theory.

Ch. 4 is on incompleteness, and the approach here is significantly more gentle than the previous chapter. Goldstern and Judah make things rather easier for themselves by adopting a version of Peano Arithmetic which has exponentiation built in (so they don't need to tangle with Gödel's β function). And they only prove a semantic version of Gödel's first incompleteness theorem, assuming the soundness of PA. The proof here goes as by showing directly that – via Gödel coding – various syntactic properties and relations concerning PA are expressible in the language of arithmetic with exponentiation (in other words, they don't argue that those properties and relations are primitive recursive and then show that PA can express all such properties/relations).

How well, how accessibly, is this done? The authors hack through eleven pages (pp. 207–217) of the arithmetization of syntax, but the motivational commentary is brisk and yet the proofs aren't completely done (the authors still leave to the reader the task of e.g. coming up with a predicate satisfied by Gödel numbers for induction axioms). So this strikes the present reader as really being neither one thing nor another – neither a treatment with all the details nailed down, nor a helpfully discursive treatment with a lot of explanatory arm-waving. And in the end, the diagonalization trick seems to be just pulled like a rabbit out of the hat. After proving incompleteness, they prove Tarksi's theorem and the unaxiomatizability of the set of arithmetic truths.

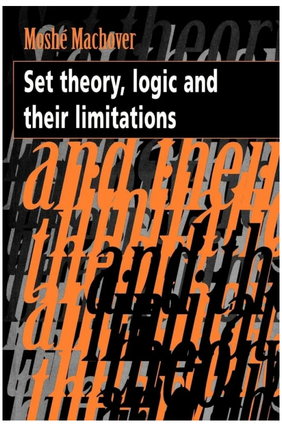
To repeat, the authors assume PA's soundness. They don't say anything about why we might want to prove the syntactic version of the first theorem, and don't even mention the second theorem which we prove by formalizing the syntactic version. So this could well leave students a bit mystified when they come across other treatments.

The book ends by noting that the relevant predicates in the arithmetization of syntax are Σ_1, and then *defines* a set as being recursively enenumerable if it is expressible by a Σ_1 predicates (so now talk of recursiveness etc. does get into the picture). But really, if you want to go down this route, this is surely all much better handled in Leary and Kristiansen's book.

◊

Summary verdict The first two chapters of this book can't really be recommended either for making a serious start on first-order logic or for revision. The third chapter could be used for a brisk revision of some model theory if you have already done some reading in this area. The final chapter about incompleteness (which the title of the book might lead you to think will be a high point) isn't the most helpful introduction in this style – go for Leary and Kristiansen (2015) instead – and on the other hand doesn't go far enough for revision/consolidatory purposes.

14 Machover 1996



Back in the day, Moshé Machover together with his colleague John Bell wrote a very substantial graduate-level text *A Course in Mathematical Logic* (see p. 18 above). Machover's later, and much shorter, *Set Theory, Logic and their Limitations* (CUP , 1996) gives us a significantly more reader-friendly introduction to some of the topics of the big book. It is based on his notes for courses given to undergraduate philosophers and mathematicians in the University of London. The style of presentation of the technical material is mathematical: but, as Machover says, "philosophical and methodological issues are often highlighted instead of being glossed over, as is quite common in texts addressed primarily to students of mathematics." So this promises to offer an introductory text very much in the style that the Beginning Mathematical Logic Study Guide favours.

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How does the story in the book unfold? Ch. 1 is a quick introduction (for those that need one) to arguments by mathematical induction. The less mathematical beginner needs get familiar with such arguments: one option is Velleman's How to Prove It; but Machover's accessible short chapter makes a good alternative.

Chs 2 to 6 (92 pp.) then provide an introduction to semi-formal set theory, getting as far as equivalents of the Axiom of Choice and the arithmetic of the Alephs (it's semi-formal in that we go beyond informal naive set theory, and are introduced to the ZFC axioms, but on the other hand we don't yet have an official formal deductive system). This is all very clearly done, I think. But arguably, for many readers Machover's treatment will fall between two stools. On the one hand, it goes further than explaining the basics of naive informal set theory taken for granted in many logic texts; on the other hand, if you are going to tackle as much material, the somewhat more expansive coverage in e.g. Enderton's introductory book will probably engender more understanding.

Next, Chs 7 and 8 (93 pp.) are on propositional and first-order logic. Now, the proof systems here for PL and FOL are pretty conventional Hilbert-style. And the syntax and semantics for FOL is also conventional, so the same letters can occur as parameters and as parts of a quantifying operator, with the consequent need to fuss about free vs bound occurrences of variables, and fuss too about substitution, variable capture etc. (Interestingly, in the Introduction to their big book, Bell and Machover do note that these rebarbative complexities – their phrase! – could have been avoided by using different symbols for free and bound variables, and they offer what strike me as somewhat feeble reasons for sticking to the Tarskian line.)

There is quite a bit of careful commentary and explanation as we go through these chapters. But my sense is that as get on to the proof of completeness for FOL, for example, the presentation becomes a notch or two less beginner-friendly than some of the alternatives. But that‘s quite a close call: these chapters should probably get a friendly mention in the Guide when I return to compare them directly to the current recommendations.

Finally, Chs 9 and 10 (81 pp.) first give a quite informal introduction to the ideas of computable functions and computable enumerability (informal in the sense that we don't get official accounts of Turing machines, register machines, partial recursive functions and the like). But it is shown – indeed as in the opening chapters of my IGT – that you can already establish a range of significant results. Then we meet the MRDP theorem, which isn't proved though its content is clearly explained.

The final chapter then turns to formal theories of arithmetic (both a series of computably axiomatised theories and True Arithmetic). With the MRDP theorem taken for granted, it is shown e.g. that True Arithmetic is not computably axiomatizable, and ultimately we reach rather strong semantic and syntactic versions of the first incompleteness theorem. So Machover's path through this material is not a usual one, and his discussion in the last two chapters is all the more illuminating for that. Definitely to be recommended, then, as supplementary reading on these topics.

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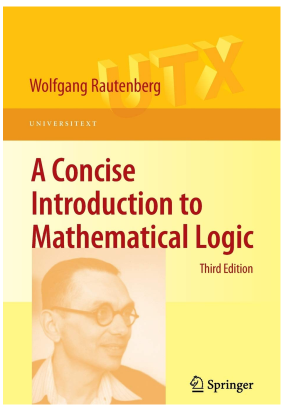
Summary verdict I can't explain why *Set Theory, Logic and their Limitations*, though sitting on my bookshelves, dropped below my radar when writing earlier versions of what became the Guide. A strange omission, it now strikes me. This always was an excellent book; no longer, perhaps a first choice on its topics, but the final chapters in particular can still be recommended.

15 Rautenberg, 1995, 2010



The first German edition of Wolfgang Rautenberg's book was published in 1995: the second edition was translated and published in English in 2006 as *A Concise Introduction to Mathematical Logic*. A slightly expanded third edition appeared in 2010 (Springer: pp. 319).

You can read the first couple of chapters of the latest edition at the author's website, linked here tinyurl.com/rautlog.

As you will see from that long excerpt, the book becomes brisk and increasingly compressed. Rautenberg says that his aim is 'to portray simple things simply and concisely'. This makes the book action-packed. After the early pages, it would certainly be tough reading as your *first* serious logic book.

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Some details Chapter 1 is a snappy treatment of propositional logic. The formal calculus of sequents offered is a close cousin of the one in Ebbinghaus, Flum and Thomas text, except where they take $\neg$ and $\vee$ to be primitive, Rautenberg has $\neg$ and $\wedge$. In both texts, given the paucity of basic operators, the respective authors are not really aiming for natural deduction in sequent form; nor are they aiming for a classical system which nicely relates to an intuitionistic subsystem. But this is all crisply done, and could make for good revision material.

Chapter 2 starts with section on structures, which will probably go too fast for those not already familiar with the ideas. Then the rest of the chapter discusses, again at pace, the syntax and semantics of FOL languages, the relevant idea of semantic consequence, and the idea of theories. It is all perfectly respectable of course, but not really recommendable as an attractive treatment. The same goes for Chapter 3, whose main focus is the completeness theorem. We get a dense account (which deals with FOL with identity and with uncountable languages from the start), and not enough intuitive motivation for my money.

Things are already pretty hard going, but then, as the Preface frankly says, "Starting from Chapter 4, the demands on the reader begin to grow. ... The density of information in the text is rather high; a newcomer may need one hour for one page.” That’s hardly an advertisement for the sort of book that appeals to me, and I rather doubt the book will overall appeal to many who aren’t mathematical masochists. Chapter 4 is a rapid look at Herbrand’s Theorem, unification, and “the foundations of logic programming”, and Chapter 5 zips through some “elements of model theory” at speed. There are surely more accessible and illuminating treatments of this material, both for a first pass through, and for higher-level consolidation/further exploration.

Chapter 6, however, is much more approachable, on recursive functions, incompleteness and undecidability, and could well be helpful (though probably not as your first encountner with these topics). The next chapter continues the incompleteness theme by starting with an unusually detailed account of how you prove the derivability conditions for PA. Then there are sections on Gödel’s second theorem and Löb’s theorem. Like Ch. 6, again useful.

But Chapter 7 (and the book) concludes with some sections giving a rather dense/opaque discussion of provability logic which I can’t recommend.

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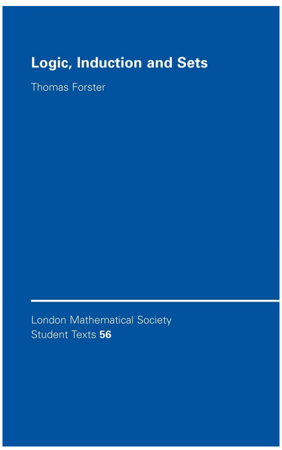
Summary verdict Concise by name, concise by nature. Too concise as an introduction, by my lights. And mostly not particularly reader-friendly for revision/consolidation purposes either. However, Chapter 6 and the first half of Chapter 7 on recursive functions and incompleteness etc. could well make for useful further reading once you already know the basics.

16 Forster, 2003



Thomas Forster's *Logic, Induction and Sets* (CUP, 2003: pp. $\mathrm{x} + 234$) is rather quirky, and some readers will enjoy it for just that reason. It is based on a wide-ranging lecture course given to mathematicians who – such being the oddities of the Cambridge tripos syllabus – at the beginning of the course already knew a good deal of maths but very little logic.

The book is very bumpily uneven in level, and often skips forward very fast, so I certainly wouldn't recommend it as an 'entry level' text on mathematical logic for someone wanting a conventionally systematic approach. But it is engaging and often intriguing.

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Some details Ch. 1 is called 'Definitions and notations' but is rather more than that, and includes some non-trivial exercises: but if you are dipping into later parts of the book, you can probably just consult this opening chapter on a need-to-know basis.

Ch. 2 discusses 'Recursive datatypes', defined by specifying a starter-pack of 'founders' and some constructors, and then saying the datatype is what you can get from the founders by applying and replying the constructors (and nothing else). The chapter considers a range of examples, induction over recursive datatypes, well-foundedness, well-ordering and related matters (with some interesting remarks about Horn clauses too).

Ch. 3 is on partially ordered sets, and we get a lightning tour through some topics of logical relevance (such as the ideas of a filter and an ultrafilter).

Chs. 4 and 5 deal slightly idiosyncratically with propositional and predicate logic, and could provide useful revision material (there's a slip about theories on p. 70, giving two non-equivalent definitions).

Ch. 6 is on 'Computable functions' and is another lightning tour, touching on quite a lot in just over twenty pages (getting as far as Rice's theorem). Again, could well be useful to read as revision, especially if you want to highlight again the Big Ideas and their interrelations.

Ch. 7 is on 'Ordinals'. Note that Forster gives us the elements of the theory of transfinite ordinal numbers *before* turning to set theory in the next chapter. It's a modern doctrine that ordinals just *are* sets, and that the basic theory of ordinals is part of set theory; and in organizing his book as he does, Forster comes nearer than most to getting the correct conceptual order into clear focus (though even he wobbles sometimes, e.g. at p. 182). However, the chapter could have been done more clearly.

Ch. 8 is called 'Set Theory' and is perhaps the quirkiest of them all – though not because Forster is here banging the drum for non-standard set theories (surprisingly given his interests, he doesn't). But the chapter is oddly structured, so for example we get a quick discussion of models of set theory and the absoluteness of Δ_0 properties *before* we actually encounter the ZFC axioms. The chapter is probably only for those, then, who already know the basics.

Ch. 9 comprises answers to some of the earlier exercises – exercises are indeed scattered through the book, and some of them are rather interesting.

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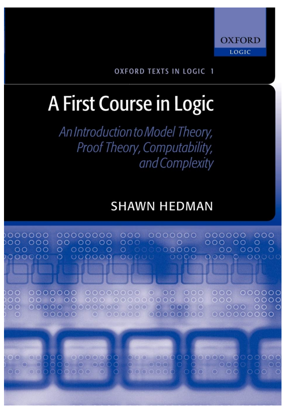
Summary verdict Different from the usual run of textbooks, not a good choice for beginners. However, if you already have encountered some of the material in one way or the other, Forster's book could very well be worth looking through for revision and/or to get some new perspectives.

17 Hedman, 2004



Shawn Hedman's *A First Course in Logic* (OUP, 2004: pp. xx + 214) is subtitled 'An Introduction to Model Theory, Proof Theory, Computability and Complexity'. So there's no lack of ambition in the coverage! And I do like the general tone and approach at the outset. So I wish I could be more enthusiastic about the book in general. But, as we will see, it is decidedly patchy both in terms of the level of the treatment of various topics, and in terms of the quality of the exposition.

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Some details After twenty pages of mostly rather nicely done 'Preliminaries' – including an admirably clear couple of pages the **P** = **NP** problem, Ch. 1 is on 'Propositional Logic'. On the negative side, we could certainly quibble that Hedman is a bit murky about object-language vs meta-language niceties. The treatment of induction half way through the chapter isn't as clear as it could be. Much more importantly, the chapter offers a particularly ugly formal deductive system. It is in fact a (single conclusion) sequent calculus, but with proofs constrained to be a simple linear column of wffs. So – heavens above! – we are basically back to Lemmon's *Beginning Logic* (1965). Except that the rules are not as nice as Lemmon's (thus Hedman's $\wedge$-elimination rule only allows us to extract a left conjunct; so we need an additional $\wedge$-symmetry rule to get from $P \wedge Q$ to Q). I can't begin to think what recommended this system to the author out of all the possibilities on the market. On the positive side, there's quite a nice treatment of a resolution calculus for wffs in CNF form, and a proof that this is sound and complete. This gives Hedman a completeness proof for derivations in his original calculus with a finite number of premisses, and he gives a compactness proof to beef this up to a proof of strong completeness.

Ch. 2, 'Structures and first-order logic' should really be called 'Structures and first-order languages', and deals with relations between structures (like embedding) and relations between structures and languages (like being a model for a sentence). I'm not sure I quite like its way of conceiving of a structure as always some $\mathcal{V}$-structure, i.e. as having an associated first-order vocabulary $\mathcal{V}$ which it is the interpretation of – so structures for Hedman are what some would call labelled structures. But otherwise, this chapter is clearly done.

Ch. 3 is about deductive proof systems for first-order logic. The first deductive system offered is an extension of the bastardized sequent calculus for propositional logic, and hence is equally horrible. Somehow I sense that Hedman just isn't much interested in standard proof-systems for logic. His heart is in the rest of the chapter, which moves towards topics of interest to computer scientists, about Skolem normal form, the Herbrand method, unification and resolution, so-called 'SLD-resolution', and Prolog – interesting topics, but not on *my* menu of basics to be introduced at this very early stage in a first serious logic course. The discussions seem quite well done, and will be accessible to an enthusiast with an introductory background (e.g. from Chiswell and Hodges) and who has read the section of resolution in the first chapter.

Ch. 4 is on 'Properties of first-order logic'. The first section is a nice presentation of a Henkin completeness proof (for countable languages). There is then a long aside on notions of infinite cardinals and ordinals (Hedman has a policy of introducing background topics, like the idea of an inductive proof, and now these set theoretic notions, only when needed: but it can break the flow). §4.3 can use the assumed new knowledge about non-countable infinities to beef up the completeness proof, give upwards and downwards LS theorems, etc., again done pretty well. §§4.4–4.6 does some model theory under the rubrics 'Amalgamation of structures'. 'Preservation of formulas' and 'Amalgamation of vocabularies': this already gets pretty abstract and uninviting, with not enough motivating examples. §4.7 is better on 'The expressive power of first-order logic'.

The next two chapters, 'First order theories' and 'Models of countable theories', give a surprisingly (I'd say, unrealistically) high level treatment of some model theory, going well beyond e.g. Manzano's book, eventually talking about saturated models, and even ending with 'A touch of stability'. This hardly chimes with the book's prospectus as being a *first* course in logic. The chapters, however, could be useful for someone who wants to push onwards, after a first encounter with some model theory.

Ch. 6 comes sharply back to earth: an excellent chapter on 'Computability and complexity' back at a sensibly introductory level. It begins with a well done review of the standard material on primitive recursive functions, recursive functions, computing machines, semi-decidable decision problems, undecidable decision problems. Which is followed by a particularly clear introduction to ideas about computational complexity, leading up to the notion of **NP**-completeness. An excellent chapter.

Sadly, the following Ch. 8 on the incompleteness theorems again isn't very satisfactory as a first pass through this material. In fact, I doubt whether a beginning student would take away from this chapter a really clear sense of what the key big ideas are, or of how to distinguish the general results from the hack-work needed to show that they apply to this or that particular theory. And things probably aren't helped by proving the first theorem initially by Boolos's method rather than Gödel's. Still, just because it gives an account of Boolos's proof, this chapter *can* be recommended as supplementary reading for those who have already seen some standard treatments of incompleteness.

The last two chapters ratchet up the difficulty again. Ch. 9 goes 'Beyond first-order logic' by speeding through second-order logic, infinitary logics (particularly $\mathscr{L}_{\omega_1\omega}$), fixed-point logics, and Lindström's theorem, all in twenty pages. This will probably go too fast for those who haven't encountered these ideas before. It should be noted that the particularly brisk account of second-order logic gives a non-standard syntax and says nothing about Henkin vs full semantics. The treatment of fixed-point logics (logics that are 'closed under inductive definitions') is short on motivation and examples. But enthusiasts might appreciate the treatment of Lindström's theorem.

Finally, Ch. 10 is on finite model theory and descriptive complexity. Beginners doing a first course in logic will again find this quite tough going.

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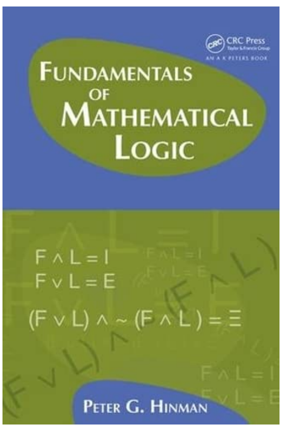
Summary verdict A very uneven book in level, with sections that work well at an introductory level and other sections which will only be happily managed by considerably more advanced students. An uneven book in coverage too. By my lights, this couldn't be used end-to-end as a course text: but in the body of the Guide, I've recommended parts of the book on particular topics.

18 Hinman, 2005



Now for a really *big* Big Book – Peter G. Hinman's *Fundamentals of Mathematical Logic* (A. K. Peters, 2005: pp. 878).

The author says the book was written over a period of twenty years, as he tried out various approaches 'to enable students with varying levels of interest and ability to come to a deep understanding of this beautiful subject'. But I suspect that you will need to be mathematically quite strong to really cope with this book: whatever Hinman's intentions for a wider readership, this is not for the fainthearted.

The book's daunting size is due to its very wide coverage rather than a slow pace – so after a long introduction to first-order logic (or more accurately, to its model theory) and a discussion of the theory of recursive functions and incompleteness and related results, there follows a *very* substantial survey of set theory, and then lengthy essays on more advanced model theory and on recursion theory. As too often, proof theory is the poor relation here – indeed Hinman is very little interested in deductive systems for logic, which don't make an appearance until over two hundred pages into the book.

Let me mention at the outset what strikes me as a pretty unfortunate global notational convention, which might puzzle casual browsers or readers who want to start some way through the book, Given the two-way borrowing of notation between informal mathematics and the formal languages in which logicians regiment that mathematics, it is good to have some way of visually distinguishing the formal from the informal (so we don't just rely on context). One common method is font selection. Thus, even in an informal context, we may snappily say that addition commutes by writing e.g. $\forall x \forall y\, x + y = y + x$; the counterpart wff for expressing this in a fully formalized language may then be, e.g., $\forall \mathsf{x} \forall \mathsf{y}\, \mathsf{x} + \mathsf{y} = \mathsf{y} + \mathsf{x}$. But instead of using sans serif or boldface for formal wffs, or another font selection, Hinman prefers using an '(informal) mathematical sign with a dot over it to represent a formal symbol in a formal language which denotes the informal object', so he'd write $\forall x \forall y\, x \dot{+} y \doteq y \dot{+} x$ for the formal wff.

As you can imagine, this convention eventually leads to really nasty rashes of dots – for example, to take a relatively tame example from p. 459, we get

$$\dot{\bigcup} x \doteq \dot{\{}z\colon \exists v[v \mathrel{\dot{\in}} x \wedge z \mathrel{\dot{\in}} v]\}$$

(note how even opening braces in formal set-former notation get dotted). This dottiness quite surely isn't a happy choice!

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Some details Hinman himself in his Preface gives some useful pointers to routes through the book, depending on your interests.

The Introduction gives a useful and approachable overview of some key notions tied up with the mathematical logician's project of formalization (and talks about a version of Hilbert's program as setting the scene for some early investigations).

Ch. 1 is on 'Propositional Logic and other fundamentals'. §§1.1, 1.3 and 1.4 are devoted to the language of propositional logic, and give the usual semantics, define the notion tautological entailment and explore its properties, giving a proof of the compactness theorem. But note, there is no discussion at all here – or in the other sections of this chapter – of a proof-system for propositional logic.

§1.2 is a rather general treatment of proofs by induction and the definition of functions by recursion (signposted as skippable at this early stage – and indeed the generality doesn't make for a particularly easy read for a section so early in the book). §§1.6 and 1.7 also cover more advanced material, mainly introducing ideas for later use: the first briskly deals e.g. with ultrafilters and ultraproducts (we get another take on compactness), and the second relates compactness to topological ideas and also introduces the idea of a Boolean algebra.

Ch. 2, 'First-order logic', presents the syntax and semantics of first-order languages, and then talks about first-order structures (isomorphisms, embeddings, extensions, etc), and proves the downward L-S theorem. We then get a general discussion of theories (thought of as sets of sentences closed under *semantic* consequence), and an extended treatment of some examples (the theory of equality, the theory of dense linear orders, and various strengths of arithmetic). There's some quite sophisticated stuff here, including discussion of quantifier elimination. But there is still no discussion yet of a proof-system for first-order logic, so the chapter could as well, if not better, have been called 'Elements of model theory'.

Ch. 3, 'Completeness and compactness', starts with a compactness proof for countable languages. Then we at last have a *very* brisk presentation of an old-school axiomatic system for first-order logic (I told you that Hinman is not interested in proof-systems!), and a proof of completeness using the Henkin construction that has already been used in the compactness proof.

We next get – inter alia – an algebraic proof of compactness for first-order consequence via ultraproducts, and a return to Boolean algebras and e.g. the

Rasiowa-Sikorski theorem (§3.3); an extension of the compactness and completeness results to uncountable languages (§3.4); and some heavy-duty applications of compactness (§3.5).

Finally in this action-packed chapter, we have some rather unfriendly treatments of higher-order logic (§3.6) and infinitary logic (§3.7).

Let's pause for breath. We are now a bit over 300 pages into the book. Things have already got pretty tough. The book is not quite a relentless march along a chain of definitions/theorems/corollaries; there are just enough pauses for illustrations and helpful remarks en route to make it a bearable. But Hinman does have a taste for going straight for abstractly general formulations (and his notational choices can sometimes be unhappy too). So as indicated in my preamble, the book will probably only appeal to mathematicians already used to this sort of fairly hardcore approach. In sum, therefore, I'd only recommend the first part of the book to the mathematically minded who already know their first-order logic and a bit of model theory; but such readers might then find it quite helpful as a beginning/mid-level model theory resource.

On we go. Next we have two chapters (almost 150 pages between them) on recursive functions, Gödelian incompleteness, and related matters. Perhaps it is because these topics are conceptually easier, more 'concrete', than what's gone before, or perhaps it is because the topics are closer to Hinman's heart, but these chapters seem to me to work better as an introduction to their topics. In particular, while not my favourite treatment, Ch. 4 is clear, very sensibly structured, and should be accessible to anyone with some background in logic and who isn't put off by a certain amount of mathematical abstraction. The chapter opens with informal proofs of the undecidability of consistent extensions of Q, the first incompleteness theorem and Tarski's theorem on the undefinability of truth (as well as taking a first look at the second incompleteness theorem). These informal proofs depend on the hypothesis that effectively calculable functions are expressible or the hypothesis that such functions are representable (we don't yet have a formal story about these functions). Unsurprisingly, given I do something in the same ball-park in my Gödel book, I too think this is a good way to start and to motivate the ensuing development. There follows, as you'd expect, the necessary account of the effectively calculable in terms of recursiveness, and then we get proofs that recursive functions can be expressed/represented in arithmetic, leading on to formal versions of the theorems about undecidable and incompleteness. This presentation takes a different-enough path through the usual ideas to be worth reading even if you've already encountered the material a couple of times before.

Ch. 5 is called 'Topics in definability' and, unlike the previous rather tightly organised chapter, is something of a grab-bag of topics. §5.1 says something about the arithmetical hierarchy; §5.2 discusses inter alia the indexing of recursive functions and the halting problem; §5.3 explains how the second incompleteness theorem is proved, and – while not attempting a full proof – there is rather more detail than usual about how you can show that the HBL derivability conditions are satisfied in PA. Then §5.4 gives more evidence for Church's

Thesis by considering a couple of other characterisations of computability (by equation manipulation and by abstract machines) and explains why they again pick out the recursive functions. §5.5 discusses ‘Applications to other languages and theories’ (e.g. the application of incompleteness to a theory like ZF which is not initially about arithmetic). These various sections are all relatively clearly done.

Pausing for breath again, we might now try to tackle Ch. 6 on set theory (whose 200 pages amount by themselves to an almost-stand-alone book). The menu covers the basics of ZF, the way we can construct mathematics inside set theory, ordinals and cardinals, then models and independence proofs, the constructible universe, models and forcing, large cardinals and determinacy. But even from the outset, this does seem quite relentlessly hard going, too short on motivation and illustrations of concepts and constructions. Dense, to say the least. The author says of the chapter that his particular mode of presentation means that ‘for each of the instances where one wants to verify that something is a class model – the intuitive universe of sets V, the constructible universe L and a forcing extension $\mathsf{m}[G]$ – . . . the proofs . . . exhibit more of the underlying unity.’ So enthusiasts who know their set theory might want to do a fast read of the chapter to see if they can glean new insights. But I can’t recommend this as a way into set theory when compared with the standard set theory texts mentioned in the Study Guide.

Ch. 7 returns to more advanced model theory for another 80 pages, getting as far Morley’s theorem. Again, if you want a more accessible initial treatment, you’ll go for Hodges’s *Shorter Model Theory*. And then why not tackle Marker’s book if you are a graduate mathematician?

Finally, there’s another equally long chapter on recursion theory. The opening sections on degrees and Turing reducibility are pretty approachable. The rest of the chapter gets more challenging but (at least compared with the material on model theory and set theory) should still be tolerably accessible to those willing to put in the work.

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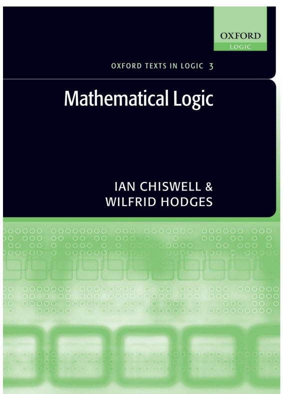
Summary verdict It is *very* ambitious to write a book with this range and depth of coverage (as it were, an expanded version of Shoenfield, forty years on – but now when there is already a wealth of textbooks on the various areas covered, at various levels of sophistication). After such a considerable labour from a good logician, it seems very churlish to say it, but the treatments of, respectively, (i) first-order logic, (ii) model theory, (iii) computability theory and incompleteness, and (iv) set theory aren’t as good as the best of the familiar stand-alone textbooks on the four areas. And I can’t see that these shortcomings are balanced by any conspicuous advantage in having the accounts in a single text, rather than a handful of different ones. Still, the text should be in any university library, as enthusiasts might well find parts of it quite useful supplementary/reference material. Chapters 4, 5 and 8 on computability and recursion work the best.

19 Chiswell & Hodges, 2007



Ian Chiswell and Wilfrid Hodges's *Mathematical Logic* (OUP, 2007: pp. 249) is very largely focused on first-order logic, only touching on Church's undecidability theorem and Gödel's first incompleteness theorem in a Postlude after the main chapters. So it is perhaps stretching a point to include it in a list of texts which cover more than one core area of the mathematical logic curriculum. Still, I wanted to comment on this book at some length, without breaking the flow of the Guide (where it is warmly recommended in headline terms), and this is the obvious place to do so.

Let me highlight three key features of the book, the first one not particularly unusual (though it still marks out this text from quite a few of the older, and not so old, competitors), the second very unusual but extremely welcome, the third a beautifully neat touch:

1. Chiswell and Hodges (henceforth C&H) present natural deduction proof systems and spend quite a bit of time showing how such formal systems reflect the natural informal reasoning of mathematicians in particular.

2. Instead of dividing the treatment of logic into two stages, propositional logic and quantificational logic, C&H take things in *three* stages. First, propositional logic. Then we get the quantifier-free part of first-order logic, dealing with properties and relations, functions, and identity. So at this second stage we get the idea of an interpretation, of truth-in-a-structure, and we get added natural deduction rules for identity and the handling of the substitution of terms. At both these first two stages we get a Hintikka-style completeness proof for the given natural deduction rules. Only at the third stage do quantifiers get added to the logic and satisfaction-by-a-sequence to the semantic apparatus. Dividing the treatment of first order logic into stages like this means that a lot of key notions get first introduced in the less cluttered contexts of propositional and/or quantifier-free logic, and the novelties at the third stage are easier to keep under control. This does make for a great gain in accessibility.

3. The really cute touch is to introduce the idea of polynomials and diophantine equations early – in fact, while discussing quantifier-free arithmetic – and to state (without proof!) Matiyasevich's Theorem. Then, in the Postlude, this can be appealed to for quick proofs of Church's Theorem and Gödel's Theorem.

This is all done with elegance and a light touch – not to mention photos of major logicians and some nice asides – making an admirably attractive introduction to the material.

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Some details C&H start with almost 100 pages on the propositional calculus. Rather too much of a good thing? Perhaps, if you have already done a logic course at the level of my intro book or Paul Teller's. Still, you can easily skim and skip. After Ch. 2 which talks about informal natural deductions in mathematical reasoning, Ch. 3 covers propositional logic, giving a natural deduction system (with some mathematical bells and whistles along the way, being careful about trees, proving unique parsing, etc.). The presentation of the formal natural deduction system is not exactly my favourite in its way representing discharge of assumptions (I fear that some readers might be puzzled about vacuous discharge and balk at Ex. 2.4.4 at the top of p. 19): but apart from this little glitch, this is done well. The ensuing completeness proof is done by Hintikka's method rather than Henkin's.

After a short interlude, Ch. 5 treats quantifier-free logic. The treatment of the semantics without quantifiers in the mix to cause trouble is very nice and natural; likewise at the syntactic level, treatment of substitution goes nicely in this simple context. Again we get a soundness and Hintikka-style completeness proof for an appropriate natural deduction system.

Then, after another interlude, Ch 7 covers full first-order logic with identity. Adding natural deduction rules (on the syntactic side) and a treatment of satisfaction-by-finite-n-tuples (on the semantic side) all now comes very smoothly after the preparatory work in Ch. 5. The Hintikka-style completeness proof for the new logic builds very nicely on the two earlier such proofs: this is about as accessible as it gets in the literature, I think. The chapter ends with a look at the Löwenheim-Skolem theorems and 'Things that first-order logic cannot do'.

Finally, as explained earlier, material about diophantine equations introduced naturally by way of examples in earlier chapters is used in a final Postlude to give us undecidability and incompleteness results very quickly (albeit assuming Matiyasevich's Theorem).

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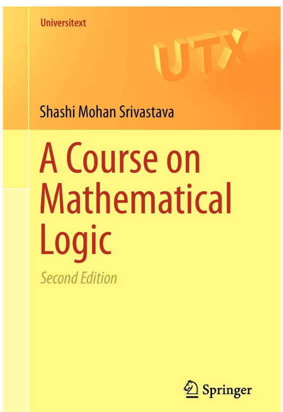
Summary verdict C&H have written a very admirably readable and nicely structured introductory treatment of first-order logic that can be warmly recommended. The presentation of the syntax of their type of (Gentzen-Prawitz) natural deduction system is perhaps done a trifle better elsewhere (Tennant's freely available *Natural Logic* gives a full dress version). But the core key sections on soundness and completeness proofs and associated metalogical results are second to none for their clarity and accessibility.

20 Srivastava, 2008, 2013



The first edition of Shashi Mohan Srivastava's *A Course on Mathematical Logic* (Springer, 2008) is notable for its brevity – propositional and first-order logic, just a little about model theory, then recursive functions and Gödelian incompleteness, all treated in just 134 pages (before the references and index). The second edition (Springer, 2013) adds some fifty pages, almost all of them in a newly inserted chapter diving much further into model theory, at a notably more advanced level than the rest of the book, rather unbalancing it.

So I'll begin by commenting on the original version of the book; then I'll add some remarks about the new material.

◊

Some details about the first edition In his Preface, Srivastava writes "Serious efforts have been made to make the book suitable for both instructional and self-reading purposes." And indeed the book strikes me as mostly admirably clear; the brevity is mainly the result of a tight focus on a selection of main topics. Though a real beginner launching herself on solo study would occasionally miss the kind of classroom chat which e.g. can help to motivate key constructions.

Chapter 1 is on the syntax, Chapter 2 on the semantics of first-order languages. The semantics is kept simple by the device of taking expanded languages with a name for every object in the domain. So that we can put $\exists v A(v)$ true in the expanded language (interpreted in a given structure, with everything named) iff, for some a in the domain, $A(i_a)$ is true (where i_a names a in the expanded language). Then a closed sentence in the original languages counts as true if true in the expanded language. Though a beginner could probably do with rather more running commentary about this approach. The chapter ends with a little about embeddings, substructres, etc. So far, so good.

Chapter 3 introduces propositional languages, with syntax, semantics, and a proof of compactness neatly done. Then there is a proof system for propositional logic, and here I have to say I don't particularly like the chosen system. Just $\neg$

and $\vee$ are primitive; we have all instances of excluded middle as axioms; and then there are four rules of inference – from A infer $B \vee A$, from $A \vee A$ infer A, from $A \vee (B \vee C)$ infer $(A \vee B) \vee C$, and from $(A \vee B)$ and $\neg A \vee C$ infer $B \vee C$. Of course it all *works* to give a sound and complete classical logic if we add the usual definitions for the other connectives. But is it elegant? Is it natural? Does it make for nice proofs? We certainly get a a messy few pages leading up to a completeness proof.

Chapter 4 gives us a proof system for FOL (with $\neg$, $\vee$ and $\exists$ primitive). The axioms are as for propositional logic plus axioms governing identity, and all instances of $A[t/x] \to \exists xA$ under the usual conditions. The rules are as for propositional logic plus the rule that from $A \to B$ with x not free in B we can infer $\exists xA \to B$. Again not exactly beautiful, and the quantifier rules are probably not expansively enough explained for beginners. After some metatheorems and a general discussion of consistency and completeness, we get a proof of completeness at the beginning of Chapter 5: this is quite nicely done. But overall not a 'best buy' as far as treatments of this essential material on FOL is concerned.

The rest of Chapter 5 (in this first edition) gives entry-level discussions of interpreting one theory in another, of extending a theory by definitions, and then turns to the Compactness Theorem and some elementary applications. There's a section on complete theories. And this helpful chapter finishes with a very brief discussion of some applications in algebra.

Chapter 6 turns to discuss recursive functions and arithmetization. Recursiveness is defined in terms of a generous supply of initial functions, composition and regular minimization. In other words, closure under primitive recursion has to be proved. I don't find this a particularly attractive or natural approach (ok, it makes the proof of results the representability of recursive functions easier, but at the price of making recursiveness a less intuitively appealing notion). Chapter 7 on incompleteness makes another disputable choice – we get Gödel only in the form "Every axiomatized, consistent extension of N [a certain formal arithmetic] is undecidable and so incomplete." Then the discussion is followed by a perhaps rather opaque treatment of the arithmetic hierarchy. Again, not the best place to start for beginners.

$\Diamond$

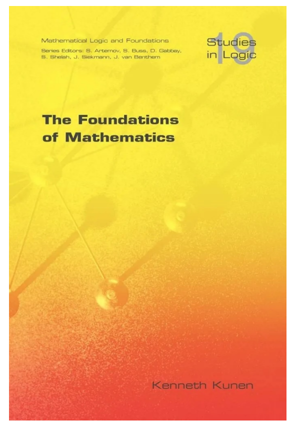
A note on the second edition's added material In the revised version, most of the old Chapter 5 gets absorbed into an expanded Chapter 4 and we have an almost entirely new Chapter 5 on 'Model Theory'. At 56 pages long, this is by far the most substantial chapter in the book. And it seems to me that the difficulty level is ratcheted up a notch or three as well. We get as far as theorems like this: *Let T be a countable, complete, ω-stable theory, with $M \vDash T$ and $A \subset M$. Then the isolated types are dense in $S_n^M(A)$.* Now, this hardly counts as really entry-level model theory, does it? So what we have in this chapter is a quite rapid-fire tour through some model-theoretic ideas which might provide useful revision material for some.

◊

Summary verdict Someone who has already met treatments of FOL and computability/incompleteness could well profit from a brisk read through the 134 pages of this first edition of this text, pausing over points of interest and thinking about the pros and cons of tackling things Srivastava's way. The added chapter in the second edition might be of use to suitably primed readers, but I can't recommend for beginners.

21 Kunen, 2009

I'm going to avert my gaze from some of the philosophical asides in Kenneth Kunen's *The Foundations of Mathematics* (College Publications, revised edn., 2009). He writes, for example,



> Presumably, you know that set theory is important. You may not know that set theory is all-important. That is:
>
> All abstract mathematical concepts are set-theoretic. All concrete mathematical objects are specific sets.
>
> Abstract concepts all reduce to set theory.

Really? *Really?* ... Well, fortunately you don't at all need to buy into such obiter dicta (or into the brief philosophical Ch. III) to find many of Kunen's technical expositions interesting and helpful.

◊

Very briefly, Ch. I (77 pp.) is on set theory, shaped by presenting the axioms of ZFC, unfolding their content and significance, getting as far as talking about ordinals and cardinals, choice, the role of the axiom of foundation, etc. As you'd hope from the author of two fine books on set theory, this is clearly done. So this chapter could suit mathematicians already a little familiar with sets-in-use from their algebra or topology courses, and/or can be recommended as a sharp and useful follow-up – one step more sophisticated but still relatively elementary – if tackled after an entry-level set-theory introduction like Enderton's.

Ch. II (100 pp.) goes on to discusses some model theory and proof theory. It treats first-order syntax and semantics, introduces a Hilbert-style proof system, and proves the completeness theorem. So far so routine, and perfectly respectable of course; but in headline terms I really didn't find the treatment of this standard material as accessible and helpful as in the previous chapter. However, the later sections of Ch. II (from §11 onwards) become more interesting, e.g. in the longer §II.17 'Definability and Absoluteness' where model theory meets set theory

Ch. IV (50 pp.) is on recursion theory. And here a key link is made with the first chapter by construing the inputs and outputs of computable functions as hereditarily finite sets. This is a neat device that puts us in the neck of the woods explored by Melvin Fitting's lovely book *Incompleteness in the Land of Sets*. Though in fact, you'll probably get much more out of tackling Kunen's interesting chapter by reading Fitting first (as well as there getting a more conventional introduction to recursion theory).

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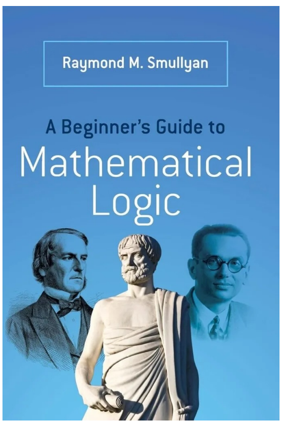
Summary verdict The first and last chapter in particular are certainly recommendable to readers with enough 'mathematical maturity' as they say.

22 Smullyan, 2014



When already in his ninety-fifth year, Raymond Smullyan published *A Beginner's Guide to Mathematical Logic* (Dover, 2014), followed three years later by a sequel *A Beginner's Further Guide to Mathematical Logic* (World Scientific, 2017).

I'm a huge admirer of Smullyan's classic books (I mean, in particular, *First-Order Logic* and the trilogy starting with *Gödel's Incompleteness Theorems*). His famed puzzle books are mightily ingenious too, though to be honest are not really so much to my own taste. Sometimes, as in his *Logical Labyrinths*, Smullyan mixes up the genres. But these late *Guides* are (relatively!) conventional in style, except for the way that significant problems (e.g. challenging you to prove theorems from hints) are scattered through the chapters, with detailed solutions at the end of chapters. The first book in particular is aimed squarely at beginners.

◊

Smullyan's *Beginner's Guide* is divided into four parts. Part I (57 pp.) is 'General Background' – something about sets, naively, but then pointers to the paradoxes; different infinite sizes of sets; mathematical induction; König's Lemma; a generalized idea of compactness; and more. The sort of thing we want a beginning logician to know at some early stage; but these chapters are perhaps a little too idiosyncratic and uneven in level and coverage to be the recommended source.

◊

Part II (68 pp.) is on 'Propositional Logic'. Ch. 5 introduces truth-tables and tautologies in a gentle and elementary way. Ch. 6 is, by contrast, a rather fast-track introduction to propositional tableaux – in just twenty pages we get signed and unsigned tableau, Smullyan's unified notation, soundness and completeness proofs, a compactness proof, and even dual tableaux. Ch. 7 introduces a familiar sort of system of axiomatic propositional logic K (closely akin to Kleene's, with four built-in connectives taken as primitive). The chapter then takes an unusual turn: it is shown that K is equivalent to a very unfamiliar axiomatic system U, and then that every propositional tableau proof can be turned into a U-proof, showing that K is complete for the usual two-valued semantics.

Part III is on 'First-Order Logic' (36 pp). Ch. 8 introduces the language of FOL, syntax and semantics, and an axiomatic system in just ten pages before the solutions to examples start – which will be surely too rushed for those who have not previous encountered a treatment. The brisk Ch. 9 is on tableaux for FOL, completeness, compactness, (downward) Löwenheim-Skolem, and more – again all very speedy.

In sum, I think Parts II and III are too uneven and in places too rushed to make for a recommendable first introduction to PL and FOL. Chapters 6 and 9 could, though, make a usable introduction to the tableaux approach to those already familiar with other versions of PL/FOL – though I do think there are rather better options at different levels.

◊

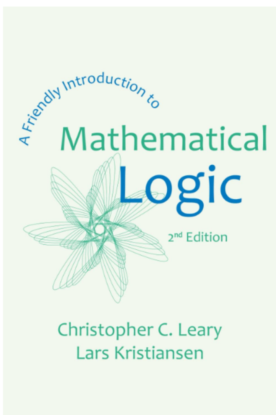
Part IV, however, strikes me as working very well, the high point of the book. The topic is 'The Incompleteness Phenomenon'. The five chapters proceed from quite abstract versions of Gödel and Tarski – harking right back to Smullyan's landmark 1959 paper 'Languages in which self reference is possible' – through to gradually more concrete semantic and syntactic versions of the first incompleteness theorems, reaching Peano Arithmetic in the penultimate chapter, with the second theorem and some related topics in the final chapter.

This is all done with over half a century of polish, with Smullyan setting more or less teasing problems as he goes along. These outstanding chapters will indeed make a quite excellent counterpoint to the more conventional, concrete-first, treatments of Gödel and Tarski, and will certainly deepen understanding. To be warmly recommended.

◊

As to Smullyan's *Further Guide*, which gets brisk, I'm inclined to think this book is really a bit too idiosyncratic to get more than a passing mention in the Logic Matters Study Guide. So I'll say no more about it here. Which isn't at all to deny that this would make for intriguing and challenging read for the interested enthusiast.

23 Leary & Kristiansen, 2015



Christopher C. Leary and Lars Kristiansen's *A Friendly Introduction to Mathematical Logic* (Milne Library 2015: pp. 364) is the second, significantly expanded, edition of a fine book originally just authored by Leary (Prentice Hall, 2000: pp. 218). The book is freely downloadable, and a printed version is available at a very attractive price; the main differences between the editions are a long new chapter on computability theory, and some 75 pages of solutions to exercises.

So how friendly is *A Friendly Introduction*? – meaning, of course, 'friendly' by the standard of logic books! I do like the tone a great deal (without being the least patronizing, it is indeed relaxed and inviting), and the level of exposition seems to me to be very well-judged for an introductory course. The book is officially aimed mostly at mathematics undergraduates without assuming any particular background knowledge. But as the Preface notes, it should also be accessible to logic-minded philosophers who are happy to work at following rather abstract arguments (and, I would add, who are also happy to skip over just a few inessential elementary mathematical illustrations).

What does the book cover? Basic first-order logic (up to the L-S theorems), the incompleteness theorems, and some computability theory. But by being so tightly focused, this book rarely seems to rush at what it *does* cover: the pace is pretty even.

The authors do opt for a Hilbertian axiomatic system of logic, with fairly brisk explanations. (If you'd never seen before a serious formal system for first-order logic this could initially make for a somewhat dense read: if on the other hand you have been introduced to logic by trees or seen a natural deduction presentation, you would perhaps welcome a paragraph or two explaining the advantages for present purposes of the choice of an axiomatic approach here.) But the clarity is indeed exemplary.

◊

Some details Ch. 1, 'Structures and languages', starts by talking of first-order languages (The authors make the good choice of not starting over again with propositional logic, but assume that most readers will know their truth-tables so just give quick revision). The chapter then moves on to explaining the idea of first order structures, and truth-in-a-structure. There is a good amount of motivational chat as we go through, and the exercises – as elsewhere in the book – seem particularly well-designed to aid understanding. (The solutions to exercises added to the new edition makes the book even more suitable for self-study.)

Ch. 2, 'Deductions', introduces an essentially Hilbertian logical system and proves its soundness: it also considers systems with additional non-logical axioms. The logical primitives are '$\vee$', '$\neg$', '$\forall$' and '$=$'. Logical axioms are just the identity axioms, an axiom-version of $\forall$-elimination (and its dual, $\exists$-introduction): the inference rules are $\forall$-introduction (and its dual) and a rule which allows us to infer φ from a finite set of premisses Γ if it is an instance of a tautological entailment. I don't think this is the friendliest ever logical system (and no doubt for reasons of brevity, the authors don't pause to consider alternative options); but it certainly is not horrible either. If you take it slowly, the exposition here should be quite manageable even for the not-very-mathematical.

Ch. 3, 'Completeness and compactness', gives a nice version of a Henkin-style completeness theorem for the described deductive system, then proves compactness and the upward and downward Löwenheim-Skolem theorems (the latter in the version 'if L is a countable language and $\mathfrak{B}$ is an L-structure, then $\mathfrak{B}$ has a countable elementary substructure' [the proof might be found *just* a bit tricky though]). So there is a little model theory here as well as the completeness proof: and you could well read this chapter without reading the previous ones if you are already reasonably up to speed on structures, languages, and deductive systems. And so, in a hundred pages, we wrap up what is indeed a pretty friendly introduction to FOL.

Ch. 4, 'Incompleteness, from two points of view' is a helpful bridge chapter, outlining the route ahead, and then defining Σ, Π and Δ wffs (no subscripts in their usage, and exponentials are atomic – maybe a footnote would have been wise, to help students when they encounter other uses). Then in Ch. 5, 'Syntactic Incompleteness – Groundwork', the authors (re)introduce the theory they call N, a version of Robinson Arithmetic with exponentiation built in. They then show that (given a scheme of Gödel coding) that the usual numerical properties and relations involved in the arithmetization of syntax – such as, ultimately, $Prf(m, n)$, i.e. m codes for an N-proof of the formula numbered n – can be represented in N. They do this by the direct method. That is to say, instead of [like my *IGT*] showing that those properties/relations are (primitive) recursive, and that N can represent all (primitive) recursive relations, they directly write down Δ wffs which represent them. This is inevitably gets more than a bit messy: but they have a very good stab at motivating every step working up to showing that N can express $Prf(m, n)$ by a Δ wff. If you want a full-dress demonstration of this result, then this is one of the most user-friendly available.

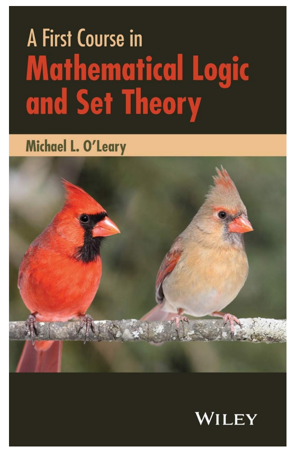
Ch. 6, 'The Incompleteness Theorems', is then pretty short: but all the groundwork has been done to enable the authors now to give a brisk but very clear presentation, at least after they have proved the Diagonalization Lemma. I did complain that, in the first edition, the proof of the Lemma was slightly too rabbit-out-of-a-hat for my liking. This edition I think notably softens the blow (one of many such small but significant improvements, as well as the major additions). And with the Lemma in place, the rest of the chapter goes very nicely and accessibly. We get the first incompleteness theorem in its semantic version, the undecidability of arithmetic, Tarski's theorem, the syntactic version of incompleteness and then Rosser's improvement. Then there is nice section giving Boolos's proof of incompleteness echoing the Berry paradox. Finally, the second theorem is proved by assuming (though not proving) the derivability conditions.

The newly added Ch. 7, 'Computability theory' starts with a very brief section on historical origins, mentioning Turing machines etc.: but we then settle to exploring the μ-recursive functions. We get some way, including the S-m-n theorem and a full-dress proof of Kleene's Normal Form Theorem (with due apologies for the necessary hacking though details) and meet the standard definition of the set $K = \{x \mid x \in W_x\}$ where W_x is the domain of the computable function with index x. The uncomputability of K is then used, in the usual sort of way, to prove the undecidability of the *Entscheidungsproblem*, to re-prove the incompleteness theorem, and in tackling Hilbert's 10th problem. This is all nicely done in the same spirit and with the same level of accessibility as the previous chapters.

◊

Summary verdict If you have already briefly met a formally presented deductive system for first-order logic, and some account of its semantics, then you'll find the opening two chapters of this book very manageable (if you haven't they'll be a bit more work). The treatment of completeness etc. in Ch. 3 would make for a nice stand-alone treatment even if you don't read the first two chapters. Or you could just start the book by reading §2.8 (where N is first mentioned), and then read the excellent ensuing chapters on incompleteness and computability with a lot of profit. *A Friendly Introduction* is indeed in many ways a very unusually likeable introduction to the material it covers, and has a great deal to recommend it.

24 O'Leary, 2016



Michael O'Leary's *A First Course in Mathematical Logic and Set Theory* (Wiley 2016, pp. 443) starts with two long chapters on propositional and predicate logic (pp. 116). There follow chapters on informal set theory, relations and functions (pp. 108), and then axiomatic set theory, ordinals and cardinals (again pp. 108). Then there is a final chapter 'Models' bringing things together, and e.g. proving the completeness of FOL (pp. 94).

The opening words of the book:

> Let us define mathematics as the study of number and space. Although representations can be found in the physical world, the subject of mathematics is not physical. Instead, mathematical objects are abstract, such as equations in algebra or points and lines in geometry. They are found only as ideas in minds.

So set theory, the topic of much of the book, being neither about number nor space, isn't mathematics? Abstract objects are ideas in minds? Well, that settles their ontological status then!

The penultimate theorem in the book:

> If the Peano axioms are provable from a consistent theory, the theory is incomplete.

Really? Let's check the definition of 'theory': p. 395 tells us that a theory is a set of sentences (there's nothing here or elsewhere about effective axiomatizability, say). True Arithmetic is a theory in that wide sense, is consistent, contains the Peano axioms and is trivially complete.

So the book starts with careless waffle and ends with a careless mistake. Which isn't too encouraging. How do things go in between?

◊

I'll be brief.

This is evidently intended very much as an entry-level text. The treatment of logic in the first two chapters is certainly no more demanding than my 'baby logic' text *Intro to Formal Logic*. It can't be recommended. For a start, the deductive system suggested for propositional logic is just an inelegant mess. The system for predicate logic involves a version of 'existential instantiation' to be deprecated (say I!). There is no real discussion in these early chapters about the semantics of FOL. Best avoided! To be honest, there's very little sense here of an author with any feel for logic, and hence this won't engender any enthusiasm in the reader.

Things do take a distinct turn for the better with chapters 3 and 4, on the elements of informal set theory. These chapters – on a relatively quick survey – look as though they could well make a nicely accessible introduction, especially for a mathematically rather weak reader. There are plenty of alternatives of course, but this is respectable enough.

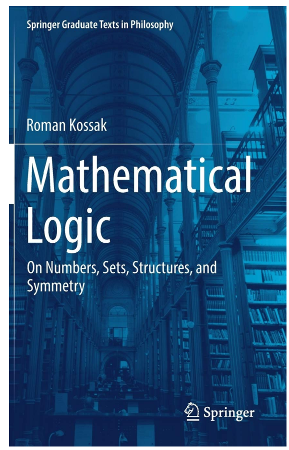
However, the following two chapters on axiomatic set theory just don't work well at all, being too short on motivations and clear explanations. Try, for one example, the section on cofinality and large cardinals and ask yourself what the beginning student is going to make of it.

And there are some odd issues of organization too. For example, the idea of the cumulative hierarchy only appears late in the very final chapter. That last chapter – also talking about FOL semantics, structures, a few model-theoretic ideas, and at long last the completeness theorem – is again not done well enough to be at all recommendable either.

Not overall a book to put on your reading list!

25 Kossak, 2018

What do you make of this?



> Think of a number, say 123. What is 123? It is a sequence of digits. To know what this sequence represents, we need to understand the decimal system. The symbols 1, 2, and 3 are digits. Digits represent the first ten counting numbers (starting with zero). The number corresponding to 123 is 1 . 100 + 2 . 10 + 3 . 1. In this representation, the number has been split into groups: three ones (units), two tens, and one hundred.

I'd worry that someone who wrote that is hopelessly confused between numerals (expressions which represent) and numbers (what the numerals represent). And just how can a number (that very thing which is represented) be "split into groups"?

Or what about this, following some examples of equinumerous collections?

> All those equinumerous collections have their individual features, but there is one thing that they all have in common. That is this ¡em¿one thing¡/em¿ that we call the size. This common feature is the size of the collection and of all other collections that are equinumerous with it. Now we can introduce the following, more formal definition: a counting number is the size of a finite collection.

So numbers are "features", i.e. *properties*? A moment ago they were things that can be split into groups! (Great-uncle Frege is not resting quietly ...)

◊

Let's put this sort of thing down to a certain arm-waving carelessness rather than confusion: still, it doesn't exactly inspire confidence in the more conceptual/philosophical remarks in Roman Kossak's *Mathematical Logic* (Springer 2018).

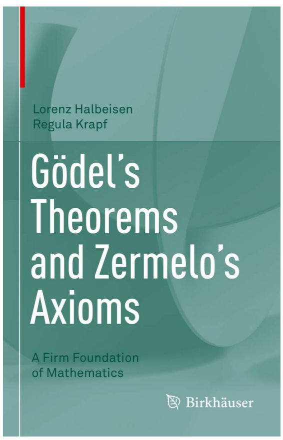
Actually, this book is mis-titled. There is little core logic here. The early chapter entitled 'First-Order Logic' is a fleeting introduction to first-order languages, too fast for real newbies, and the idea of a formal deductive system is only mentioned, and then without elaboration, at p.132 (and the book has just 155 pages before the final summary chapter and the appendices start). What the book is in fact centrally about is signalled by its subtitle: "On Numbers, Sets, Structures, and Symmetry".

As Kossak says in his final summary, his "aim in this book was to explain the concept of mathematical structure, and to show examples of techniques that are used to study them. It would be hard to do it honestly without introducing some elements of logic and set theory." The examples of structures are near all numerical ones. So Part I is an introduction to the construction of the integers, rationals and reals from the naturals, and a lightning tour of some of the presupposed set theory. This is done in with a fair amount of motivational chat, so some credit for that. But I still think the beginner would be notably better off reading one of the usual introductions to these ideas in elementary set theory books like Enderton's or Goldrei's.

Charitably, the author is rushing on to get to what really does interest him, the book's distinctive content in the seventy-odd pages of Part II. And this Part, to get much more positive, is a very approachable introduction to some simple model theoretic ideas, but taking a rather different route than in some more familiar texts. So Kossack explores some of the first-order definable features of various structures defined over the natural numbers, the integers, the rationals, the reals, and the complex numbers, and he nicely brings out some of the perhaps unexpected complications. He helps himself to the compactness theorem and e.g. the Tarski–Seidenberg theorem (which are not proved) to give partial demonstrations of various results. And along the way, the reader is introduced not only to basic notions like that of an elementary extension but also somewhat more sophisticated ideas like being a minimal structure. This is done with a light touch, helpful examples, and again a good amount of motivational chat. I'm not sure that some of the sketched proofs are quite as clear as they could be, and there can be philosophical wobbles in the commentaries. But this Part of Kossack's book does, I think, get across some basic model theoretic ideas without too many tears, in an unusually accessible way, making connections that aren't often brought out; and (those wobbles apart) I enjoyed it and learnt from it.

Since this book, however, Kossak has published his excellent *Model Theory for Beginners: 15 Lectures* (College Publications, 2021). You should probably read that instead!

26 Halbeisen & Kraft, 2020



A standard menu for a first mathematical logic course might be something like this: (1) A treatment of the syntax and semantics of FOL, presenting a proof system or two, leading up to a proof of a Gödel's completeness theorem (and then a glance at e.g. the compactness theorem and some initial implications). (2) An introduction to formal arithmetic, a little about computability, with Gödel's incompleteness theorems a highlight. (3) A modest amount of set theory, looking e.g. at the way number systems including the reals can be constructed in set theory; a first encounter with cardinals, ordinals and Choice; then the formalization of set theory in ZFC. With all this leading to an emerging sense of (4) the limitations of first-order theories and the ubiquity of non-standard models.

So an attractive, accessible, relatively short, book covering Gödel's Theorems (completeness/incompleteness) and Zermelo's Axioms (and why we need a bit more than Zermelo's original proposals) could indeed have a grateful readership.

But Lorenz Halbeisen and Regula Krapf's promisingly titled book (published by Birkhäuser in 2020) is not that book. It does tick the box of being relatively short (x + 236 pp.). However – to be blunt about it – this is not particularly reader-friendly, often unnecessarily hard going, and there are much better options, particularly for self-study.

◊

Part I introduces FOL, initially in Hilbert-style (with a selection of axioms presented in a take-it-or-leave-it spirit). Then we get what is advertised as a natural deduction system, though by p. 24 it is beginning to look more like a sequent calculus (in a way that could confuse the reader). On p. 30 we get a trivial syntactic result labelled as the compactness theorem (in a way that could again confuse the reader who was seen standard talk of compactness elsewhere). Then we get a pretty messy introduction to the semantics of FOL.

Part II proves the completeness theorem (for countable signatures), done Henkin-style. Somehow the rather neat elegance that such proofs can have is quite lost in the telling: would the student new to this come away with a good sense of the fundamental ideas? – I doubt it.

Part III starts with a short chapter on standard and non-standard models of PA, before actually looking in the next chapter at how arithmetic can be done in PA. There follows a twenty-page chapter hacking through the arithmetization of syntax for PA: ok, this is necessarily a messy business, but it is certainly done more accessibly and more attractively elsewhere.

Then we get the key chapter on the first incompleteness theorem. Once more, I can't say that this is done in a reader-friendly way. The diagonalization lemma is plucked out of the sky; and there's careless talk of the kind we warn our students against – the lemma "allows us to make self-referential statements, i.e. for a formula φ with one free variable it provides a sentence σ_φ which states 'I have the property φ'. " No it doesn't. And then the initial proof of incompleteness on p. 112 will puzzle readers who've seen (or are about to see) other presentations – how do we get that PA doesn't prove a fixed point of $\neg\mathsf{prv}$ without mentioning ω-consistency?

Moving on, we do get a full-dress eight-page proof of the second incompleteness theorem for PA, and in particular of the third derivability condition, i.e. it's shown that PA proves $prv(\ulcorner\phi\urcorner) \to prv(\ulcorner prv(\ulcorner\phi\urcorner)\urcorner)$. It probably sounds ungrateful, but is this really what we want or need in an introductory book? Only if particularly nicely done, which it isn't.

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Part IV is on set theory, starting with a twenty page chapter on the axioms of ZFC, but also explaining the set-theoretic definitions of functions and relations, saying something about Choice, and introducing ordinals and cardinals. All surely far too fast for the student for whom this really is all new.

The remaining chapters are on models of set theory, standard and non-standard; on ultrafilters and ultra products (and so completeness for FOL with uncountable signatures); briefly again on non-standard models of PA; and then on real numbers, non-standard models of the reals, and a little on non-standard analysis. My sense is that these final chapters – which are worthwhile reading for someone who has got the basics nailed down elsewhere – is where the authors' real interest lies.

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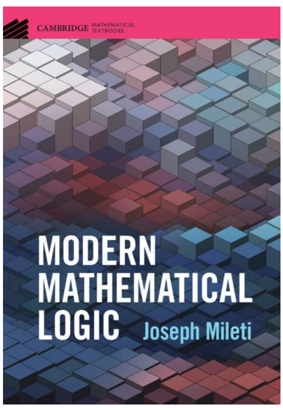
Summary verdict You can cheerfully ignore what's gone before (there are many better options for introducing their material). But students with enough background might find the last four chapters, some 50 pages, profitable.

September 2024

27 Mileti, 2023



Joseph Mileti's *Modern Mathematical Logic* (CUP 2023, 502 pp.) is announced as begin aimed at advanced undergraduates or beginning graduates.[1]

Despite the title, the coverage is old-school and the approach thoroughly conventional. Mileti starts with basic first-order logic (though there's no real proof theory). Then there's a little model theory, entry-level axiomatic set theory, some computability theory, and the book ends with a treatment of incompleteness.

A familiar menu, then: how well is it served it up? There is friendly signposting and some nice turns of phrase. But ...

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Some details Chapter 2, the first substantial chapter, is thirty pages on 'Induction and Recursion'. We get a pretty dense treatment of what Mileti calls 'generating systems', three different ways of defining a set of generated whatnots, proofs that these definitions come to the same, then a criterion for free generating systems, a proof that we can do recursive definitions over the free systems, and so on. This is all done in what strikes me as a rather heavy-handed way which could be pretty off-putting as a way of starting out. Many students, I would have thought, will just feel they have been made to labour unnecessarily hard at this point for small returns. And when the very general apparatus is applied e.g. in the next chapter to prove, e.g., unique parsing results, I don't think that what we get is more illuminating than a more local argument. (I suppose my pedagogic inclination in such cases is to motivate a general proof idea by proving an interesting local case first and then, at an appropriate point later, saying "Hey, we can generalize ...".)

Chapter 3, the next fifty pages, is on propositional logic. A minor complaint is that the arrow connective is initially introduce in the preface as meaning 'im-

[1]Mileti says he assumes familiarity with some basic abstract algebra; however, this seems perhaps more needed to best appreciate some illustrative examples rather than as necessary background for grasping core content.

plies' (oh dear), and then we get not a word of discussion of the truth-functional treatment of the connective (unless my attention flickered). But my main beef here is with the chosen formal proof system. This is advertised as natural deduction, but it is a sequent system, where on the left of sequents we get sequences rather than sets (why?). And although the rules are set out in a way that would naturally invite tree-shaped proofs, they are actually applied to produce linear proofs (why?). Moreover, the chosen rule-set is not happily motivated. We have disjunctive syllogism rather than a proper vE rule; double negation elimination is called ¬E (so intuitionistic logic doesn't have negation elimination?– Mileti's system doesn't have a nice inituitionistic subsystem). OK: Mileti isn't going to be interested in proof theory; but he should at least have chosen a modern(!) proof system with proof-theoretic virtues!

We then get the sort of propositional completeness proof that (a) involves building up a maximal consistent set starting from some given wffs by looking at every possible wff in turn to see if it can next be chucked into our growing collection while maintaining consistency, rather than the sort of proof that (b) chucks in simpler truth-makers only as needed, Hintikka style. We are not told what might make the Henkin strategy better than the more economical Hintikka one.

Perhaps the best/most interesting thing in this chapter is the final section (and the accompanying exercises) on compactness for propositional logic, which gives a nice range of applications.

Chapter 4 is on 'First-order logic: languages and structures' – so some 40 pages on basic semantics. Chapter 5 is on 'Relationships between structures' – another longish chapter, 35 pages on substructures, homomorphisms between structures, embeddings, and the like. Chapter 6 , 'Implications and compactness', is an even longer chapter – some 48 pages introducing a proof system for FOL and proving soundness and completeness, then drawing out some consequences of compactness, before going on to talk about theories framed in a first-order language, with a substantial final section on random graphs.

In headline terms: I found the basic treatment of the semantics, and again of the formal proof-system and completeness for FOL, pretty unattractive. On the other hand, the more model-theoretic Chapter 5, and the second half of Chapter 6 strike me as notably more readable.

In just a bit more detail, we get a highly conventional story about the syntax of FOL languages. In particular, the same symbols are recruited for double duty, as part of the construction of a quantifier operator, and for use as parameters/temporary names. Of course, this means we have to fuss about rules for distinguishing free from bound occurrences of variables, and fuss at length about avoiding unwanted variable capture when substituting terms for variables (§4.4 on substitution is no less than eleven rather dense pages long). Why do things this way? It's only ninety years since Gentzen taught us how to do better, in ways that have become more and more familiar as modern proof-theorists spread the word!

In the middle of Chapter 4, though, there is a nice short first section on definability. Issues of definability and related topics about what classes of structures can be captured by which languages, and so on, are then taken up in the next chapter – which ends with a nice section §5.5 which introduces the Tarski-Vaught test and shows how to get from there to a version of the downward L-S theorem for a countable language. §4.3 and Chapter 5 could I think be tackled standalone by someone who knows some basic FOL from other sources; and these sections do work pretty well.

After a section defining semantic entailment for FOL, Chapter 6 introduces a deductive system for quantificational logic, far too briskly (it seems to me) to be of much use to anyone who is encountering one for the first time. And the soundness and completeness results are done no more attractively than for the earlier propositional logic case. I can't really recommend these sections. But then §6.4 on applications of compactness and §6.5 on theories are nice (and the concluding section on random graphs is an interesting bonus).

Chapter 7 is titled 'Model theory'. Of the five sections, the first three can't be recommended. In particular, §7.2 makes unnecessarily heavy weather of that fun topic, nonstandard models of arithmetic and analysis. By contrast, I thought §7.4 on quantifier elimination did a notably better-than-often job at explaining the key ideas and working through examples. §7.5 on algebraically closed fields worked pretty well too.

And now we get two chapters on set theory, together amounting to almost a hundred pages. There's a major oddity. The phrase 'cumulative hierarchy' is never mentioned: nor is there any talk of sets being found at levels indexed by the ordinals. The usual V-shaped diagram of the universe with ordinals running up the spine is nowhere to be seen. I do find this very strange – and not very 'modern' either! There are minor oddities too. For example, the usual way of showing that the Cartesian product of A and B (defined as the set of Kuratowski pairs $\langle a, b\rangle$) is a set according to the ZFC axioms is to use Separation to carve it out of the set $\mathcal{P}(\mathcal{P}(A \cup B))$ in the obvious way. Mileti instead uses an unobvious construction using Replacement. Why? A reader might well come away from the discussion with the impression that Replacement is *required* to get Cartesian products and hence all the constructions of relations and functions which depend on that.

So: Chapter 8 gives us ZFC, and the usual sort of story about how to develop arithmetic and analysis in set theory. The mentioned oddities apart it is generally OK: but the recommendations for entry-level set theory in the Study Guide do the job better and in a friendlier way. However I should mention that, at the end of the chapter, §8.7 on models, sets and classes, does explain the role of class talk rather nicely.

Chapter 9 is on ordinals, cardinals, and the axiom of choice; and I thought this chapter worked comparatively well. Finally in this group, Chapter 10 is much shorter, just two sections on "Set-theoretic methods in model theory". The first, just four pages, is on sizes of models; and then the second is an opaque and to my mind misjudged ten pages on ultraproducts.

We are now on the home straight ... only 117 pages to go. The last two long chapters are on 'Computable sets and functions' and 'Logic, computation, and incompleteness'.

In broad-brush terms, the content is pretty much the sort of thing you could predict. So Chapter 11, some seventy pages, introduces the primitive recursive functions, shows they are not all the intuitively computable functions, and so goes on to discuss partial recursive functions. Then we get a machine model of computation, with Mileti choosing URM machines over Turing machines. We find out that the URM computable partial functions are just the partial recursive functions, and there is some sensible discussion of the Church-Turing Thesis. The chapter concludes by looking at computably enumerable (but perhaps not computable) sets.

Then Chapter 12 starts by talking about coding expressions and deductions, and about arithmetic definability. §12.3 shows that the set of true sentences of formal first-order arithmetic is undecidable. Mileti then starts looking at Robinson Arithmetic in particular and shows that it can represent computable functions. The final section of the book gives us a proof of incompleteness.

So these final two chapters cover material which is already beautifully covered in some classic books from e.g. the early editions of Boolos and Jeffrey onwards. To be sure, these chapters are perfectly respectable, probably the best in the book, and Mileti can write with an engaging turn of phrase. But are they particularly attractively done, especially accessible, splendidly clear, plainly to be preferred to the existing entry-level recommendations in the Study Guide? I rather think not.

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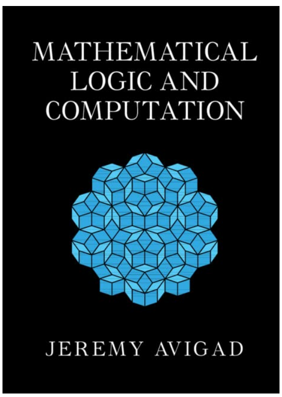
Summary verdict Some parts of this book can be recommended for supplementary reading for self-study. But to be frank, I'm still not quite sure what the *point* of Mileti's text is. The title rather belies the content – what's so 'modern' here? The treatments of the various topics do usually seem thoroughly conventional and often even rather old-school. And I'm not persuaded that – sixty years on from Mendelson! – there is still any special additional virtue in having core FOL, some model theory, set theory, and some computability theory all done within a single set of covers, a benefit that makes the book worth more that the sum of its parts. So, I'm afraid I can't jump to join in the chorus of rather extravagant praise printed at the front of the book. (Though equally, if you do want to really, *really*, insist on having just one single text to back up a lecture course covering the whole of the traditional menu for a wide-ranging multi-semester mathematical logic course, I guess this could be your best option at its level.)

August 2023

28 Avigad, 2023



Jeremy Avigad's *Mathematical Logic and Computation* (CUP 2023, pp. 513) is – despite the author's intentions – not really a book for beginners. And it makes for a bumpy ride, with chapters (or sections within chapters) at notably different levels of compression and difficulty. But for all that it is an impressive sourcebook with some particularly useful episodes.

The first seven chapters, amounting to 190 pages, form a book within the book, on core FOL topics but with an unusually and distinctively proof-theoretic flavour. The rest of the book is about arithmetic and computability, with nods towards type theory. I'll take these two parts in turn.[1]

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Some details on FOL Chapter 1 is on 'Fundamentals', aiming to 'develop a foundation for reasoning about syntax'. So we get the usual kinds of definitions of inductively defined sets, structural recursion, definitions of trees, etc. and applications of the abstract machinery to defining the terms and formulas of FOL languages, proving unique parsing, etc., all done in a quite hard-core way.

Then Avigad has two substantial and wide-ranging chapters on propositional logic. So §2.1 quickly reviews the syntax of PL (with $\wedge, \vee, \rightarrow, \bot$ as basic – so negation has to be defined by treating $\neg A$ as $A \rightarrow \bot$). §2.2 presents a Hilbert-style axiomatic deductive system for minimal logic, which is augmented to give systems for intuitionist and classical PL. §2.3 says more about the provability relations for these three logics. §2.4 then introduces natural deduction systems for the same three logics, and outlines proofs that they prove the same deductions. §2.5 notes some validities in the three logics and §2.6 is on normal forms in classical logic. §2.7 then considers e.g. the Gödel-Gentzen double-negation translation between intuitionist and classical logic. Finally §2.8 takes a very brisk look at other sorts of deductive system, and issues about decision procedures.

[1]This book note is cut down from an even longer blog post here.

To continue, Chapter 3 is on semantics. We get the standard two-valued semantics for classical PL, along with soundness and completeness proofs, in §3.1. Then we get interpretations in Boolean algebras in §3.2. Next, §3.3 introduces Kripke semantics for intuitionistic (and minimal) logic. §3.4 gives algebraic and topological interpretations for intuitionistic logic. And the chapter ends with a pretty challenging §3.5, 'Variations', introducing a generalized Beth semantics.

As you can see, a *lot* is going on here, so this is not I think for real beginners. But for someone coming to the book who already has enough logical background, these chapters – perhaps initially minus their last sections – should bring a range of technical material into a nicely organised story in a very helpful way.

The next two chapters are on the syntax and proof systems for FOL – in three flavours again, minimal, intuitionstic, and classical – and then on semantics and a smidgin of model theory. Again, things proceed at considerable pace.

Broadly following the pattern of the two chapters on PL, in §4.1 we find a brisk presentation of FOL syntax.§4.2 presents axiomatic and ND proof systems for the quantifiers, adding to the systems for PL in the standard ways. §4.3 deals with identity/equality and says something about the 'equational fragment' of FOL. §4.4 says more than usual about equational and quantifier-free subsystems of FOL, noting some (un)decidability results. §4.5 briefly touches on prenex normal form. §4.6 picks up the topic (dealt with in much more detail than usual) of translations between minimal, intuitionist, and classical logic. §4.7 is about how – when we can prove $\forall x \exists ! y A(x, y)$ – we can add a function symbol f such that $f(x) = y$ holds when $A(x, y)$. Finally, §4.8 treats two topics: first, how to mock up sorted quantifiers, and how to deal with partially defined terms.

Then, turning to semantics, §5.1 is the predictable story about full classical logic with identity, with soundness and completeness theorems, all crisply done. §5.2 tells us more about equational and quantifier-free logics. §5.3 extends Kripke semantics to deal with quantified intuitionistic logic. We then get algebraic semantics for classical and intuitionistic logic in §5.4 (so, as before, Avigad is casting his net much more widely than usual). The chapter finishes with a fast-moving 10 pages on model theory. §5.5 deals with some (un)definability results, and talks briefly about non-standard models of true arithmetic. §5.6 gives us the L-S theorems and some results about axiomatizability.

So that's again a great deal packed into these chapters. And at a sophisticated level too. So the same judgement applies, I think, as to the earlier chapters on PL: the chapters on FOL contain a lot of very good material for someone already on top of the basics, and wanting to consolidate and expand their knowledge, but are probably not the place to start.

The book continues, then, with Chapter 6 on Cut Elimination. And the order of explanation here is, I think, interestingly and attractively novel.

Things begin in a familiar way. §6.1 introduces a standard sequent calculus for (minimal and) intuitionistic FOL logic without identity. §6.2 then, again in the usual way, gives us a sequent calculus for classical logic by adopting Gentzen's device of allowing more than one wff to the right of the sequent sign. But then

Avigad notes that we can trade in two-sided sequents, which allow sets of wffs on both sides, for one-sided sequents where everything originally on the left gets pushed to the right of sequent side (being negated as it goes).

So in §6.2 we are introduced to a classical calculus using such one-sided, disjunctively-read, sequents (we can drop the sequent sign as redundant) – and it is taken that we are dealing with wffs in 'negation normal form', i.e. with conditionals eliminated and negation signs pushed as far as possible inside the scope of other logical operators so that they attach only to atomic wffs. This gives us a very lean calculus. There's the rule that any $\Gamma, A, \neg A$ with A atomic counts as an axiom. There's just one rule each for $\wedge$, $\vee$, $\forall$, $\exists$. There also is a cut rule, which tells us that from Γ, A and $\Gamma, {\sim}A$ we can infer Γ (here ${\sim}A$ is notation for the result of putting the negation of A into negation normal form).

And Avigad now proves twice over that this cut rule is eliminable. So first in §6.3 we get a semantics-based proof that the calculus without cut is already sound and complete. Then in §6.4 we get a proof-theoretic argument that cuts can be eliminated one at a time, starting with cuts on the most complex formulas, with a perhaps exponential increase in the depth of the proof at each stage. And the sparse one-sided calculus does make for a nicely minimal context in which to run a bare-bones proof-theoretic argument for the eliminability of the cut rule, where we have to look at a very small number of different cases in developing the proof instead of having to hack through the usual clutter. I do have to report though that, to my mind, Avigad's compressed mode of presentation doesn't really make the proof any more accessible than usual.

To continue (and I'll be briefer) §6.5 looks at proof-theoretic treatments of cut elimination for intuitionistic logic, and §6.6 adds axioms for identity into the sequent calculi and proves cut elimination again. §6.7 is called 'Variations on Cut Elimination' with a first look at what can happen with theories other than the theory of identity when presented in sequent form. Finally §6.8 returns to intuitionistic logic and (compare §6.5) this time gives a nice semantic argument for the eliminability of cut, going via a generalization of Kripke models.

This is all very good stuff. But I hope it doesn't sound too ungrateful to say that a student new to sequent calculi and cut-elimination proofs would still do best to read the initial chapters of Negri and von Plato (for example) first, if they are later to be able get a lively appreciation of the discussion here.

Following on from the very interesting Chapter 6 on cut-elimination, Avigad has one further chapter on FOL, Chapter 7 on 'Properties of First-Order Logic'. There are sections on Herbrand's Theorem, on the Disjunction Property for intuitionistic logic, on the Interpolation Lemma, on Indefinite Descriptions and on Skolemization. This does nicely follow on from the previous chapter, as the proofs here mostly rely on the availability of cut-elimination. I'm not going to dwell on this chapter, though, which I think most readers will find pretty hard going. Hard going in part because, apart from perhaps the interpolation lemma, it won't this time be obvious from the off what the point of various theorems are.

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Some details on computation, arithmetic, etc. I'm now going to be speeding up somewhat. So I won't pause over Ch. 8 on primitive recursive functions, which is full of interesting technical details – but once again I suspect that many readers will find this chapter most useful if they have already seen a first introduction.

But while Ch. 8 might be said to be relatively routine, Ch. 9 is anything but. It is the most detailed and helpful treatment of Primitive Recursive Arithmetic that I know. Avigad first presents an axiomatization of PRA in the context of a classical quantifier-free first-order logic. We have a symbol for each p.r. function – and we can think of these added iteratively, so as each new p.r. function is defined by composition or primitive recursion from existing functions, a symbol for the new function is introduced along with its appropriate defining quantifier-free equations in terms of already-defined functions. We also have a quantifer-free induction rule: from $A(0)$ and $A(x) \to A(Sx)$, infer $A(t)$ for any term. §§9.1–9.3 explore this version of PRA in some detail, deriving a lot of arithmetic.

Then the next two sections very usefully discuss two variant presentations of PRA. §9.4 enriches the language and the logic by allowing quantifiers, though induction is still just for quantifier-free formulas. It is proved that this is conservative over quantifier-free PRA for quantifier-free sentences. By contrast §9.5 weakens the language and the logic by removing the connectives, so all we are left with are equations, and we replace the induction rule by a rule which in effect says that functions satisfying the same p.r. definition are everywhere equal. All this is done in a quite exemplary way, I think.

The following Ch. 10 is the longest in the book, some forty two pages on 'First-Order Arithmetic'. Or rather on arithmetics, plural – for as well as the expected treatment of first-order PA, with nods to Heyting Arithmetic, there is also a perhaps surprising amount here about subsystems of classical PA.

In more detail, §10.1 briefly introduces PA and HA. You might expect next to get a section explaining how PA can be in fact seen as extending the PRA which we've just met. But we have to wait until §10.4 to get the story about how to define p.r. functions using some version of the beta-function trick. In between, there are two longish sections on the arithmetical hierarchy of wffs, and on subsystems of PA with induction restricted to some level of the hierarchy.

The material here is all stuff that is very good to know. You won't be surprised by this stage to hear that the discussion is a bit dense in places; but up to this point it should all be pretty manageable.

However, the chapter ends with another ten pages proving (1) that $\mathrm{I}\Sigma_1$ is conservative over PRA for Π_2 formulas; (2) Parikh's Theorem, (3) that so-called $\mathrm{B}\Sigma_{n+1}$ is conservative over $\mathrm{I}\Sigma_n$ for Π_{n+2} wffs. I am not sure what gives these results enough interest at this level to labour through them.

So moving on, Ch. 11 is on computability. §§11.1–11.6 are a fast-track introduction to the basics of the theory of partial recursive functions together with a look at Turing machines. We get to Rice's theorem in ten pages, which tells you how very fast things go. Still, at its level, it is very clear and nicely motivated, so we can certainly put this down as useful revision material.

§11.7, however, takes an even faster six-page look at the untyped lambda calculus. I found this section very strange: I really do have to wonder what someone genuinely new to the topic could actually get out of this compressed presentation. And then, after a section on relative computability, Ch. 11 ends with a pretty hard-core section on computability and infinite binary trees.

We get back to basics for most of Ch. 12 on undecidability and incompleteness. My sense is that there is rather more motivational chat here that in some episodes in the book, and I do think §§12.1–12.4 really do work very nicely. Again, §§12.1–12.4 can be recommended reading in the Study Guide. The final section of the chapter, §12.5, however, is a terse discussion of theories that can be shown to be complete by arguments using quantifier elimination results – I found the mode of presentation to be distinctly less reader-friendly again.

There is a considerable jolt upwards in level of difficulty as we move to the next two chapters, though. Ch. 13 is on the simply typed lambda calculus, combinatory logic and more: the discussion is too compressed to be very useful, I would judge. Most readers brand new to the topic would surely struggle. Very much the same goes for Ch. 14 on realizability, the Dialectica interpretation and more, though this is interesting and helpful reading if you already know enough.

Ch. 15 is on second-order logic and arithmetic. Once again, this goes at a cracking pace, and would be pretty tough for someone's first exposure to SOL. However, these early sections might well make for higher-level revision, if you'd first taken things more slowly in some other presentation. But heavens, by §15.6 we are talking about the second-order typed lambda calculus ...

Ch. 16 is on subsystems of second-order arithmetic. I'm all for seeing this topic covered in a text like this. Though – stuck record though I am proving to me – I find myself again wanting to say that this surely all goes a bit too densely for a first exposure to the topic. Though yes, a good read perhaps if you have already tackled the more accessible first chapter of Simpson's now classic book.

Finally, MLC has a concluding chapter on 'Foundations' – a very speedy 16 page tour through simple type theory, set theory, dependent type theory and more. This sits rather oddly with what's gone before.

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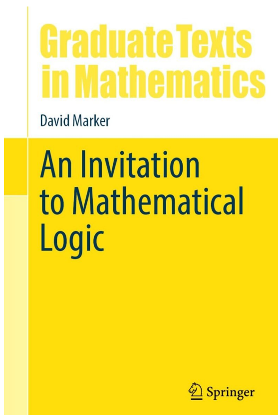
Summary verdict I have kept remarking on the differences in level (sometimes quite radical) between different chapters, and quite often between sections within a chapter. They do make me wonder. Does this book in fact have a number of different archaeological layers, with different parts having their ultimate origins in handouts for differently paced, different level courses? I do wonder! It certainly makes, as I said before, for a prettyuneven ride. But there is, for all that, a great deal of highly interesting material here in MLC: you'll just need to be primed to a suitable level (different for different episodes) to really appreciate it.
August 2023

29 Marker, 2024



David Marker, the author of what has become a modern classic on model theory, has now published *An Invitation to Mathematical Logic* (Springer, 2024). “My goal was to write a text for a one-semester graduate-level introduction to mathematical logic, one that I would have liked to learn from when I was a student, and one I would like to teach from as a professor.”

Part I of the book, ‘Truth and Proof’ is on first-order logic and theories and the structures for interpreting them. Part II is on ‘Elements on Model Theory’. Part III is on ‘Computability’ and Part IV discusses ‘Arithmetic and Incompleteness’. (The book doesn’t discuss set theory.)

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In a little more detail, Part I (64 pp.) has four chapters. Ch. 1, ‘Languages, Structures, and Theories’ provides a brisk introduction, but one which is really rather short on motivations and explanations.

A quite trivial but characteristic example: we are flatly told that $\varphi \rightarrow \psi$ is an abbreviation of $\neg\varphi \vee \psi$, take it or leave it, end of story: so much for calming the common student discomfort – graduate student or otherwise – with the conventional treatment of the conditional! Again a standard Tarksi-style semantics for the quantifiers is bluntly stated, without comment, take it or leave it: fine if you’ve met the idea before, but the student quite new to this might reasonably ask for just a bit more by way of explication.

We get the same briskness in Ch. 2, ‘Embeddings and Substructures’. Then the short Ch. 3 introduces one proof system for FOL, a sequent calculus, in which proofs are simple linear arrays, Hilbert-style. A perfectly serviceable system, but there’s no hint at all about different ways of doing things.

Ch. 4 proves completeness, in places a bit laboriously. Marker does bring out nicely why the story goes a bit differently for countable languages and uncountable languages (needing Zorn’s Lemma or an equivalent in the second case). But on the other hand – the student reader might again reasonably ask – given that all the earlier examples of FOL languages involve small finite non-logical vocabularies, exactly why might we care about the uncountable case? I don't think we are told.

These chapters are of course all done perfectly respectably, and there are nice episodes: but just how inviting are they to the reader quite new to the area? I, for one, didn't find them particularly so, and I at least had the advantage of already knowing what was supposed to be going on. Marker tends to really short-change the beginner when it comes to those useful orientating sentences or two which can be so helpful (the classroom asides, the "look at it this way" guides). And relatedly, some of his proofs can leave the reader to distinguish the interesting moves from the bits where we are just joining-up-the-dots.

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Part II (71 pp.) again has four chapters. Ch. 5 is on compactness (introduced as a simple consequence of completeness), starting with some elementary applications but soon turning to examples you'll need more mathematical background to understand. Ch. 6 is a somewhat dense introduction to ultraproducts, giving us another proof of compactness. Ch. 7 begins on the basic idea of quantifier elimination; but soon, in Ch. 8, we are into fairly hardcore algebraic applications – fine for the graduate pure mathematicians with some serious algebra under their belt who are perhaps Marker's core intended audience, but again not done invitingly enough (say I) to draw in other readers whose prime interests are more logical.

I hasten to add this is not a bad book, and Parts I and II could indeed make useful parallel/additional reading to the Study Guide's prime recommendations on the topics of these two Parts, useful for someone who likes Marker's style and who wants to work beyond a first introduction. But not, to my mind, the place to start.

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Part III of Marker's book gives us a 63 pp. introduction to the theory of computability. Ch. 9 explores models of computation, first very briskly introducing unlimited register machines. We next meet primitive recursive functions, and then the partial recursive functions. It is then proved that the partial recursive functions are exactly those partial functions computable by a register machine; and we get a bit more evidence for Church's Thesis by noting that machines with random access memory won't compute more. The chapter – under 20 pages before the Exercises start – ends with a very quick glance at Turing machines.

So this all proceeds at a breathless pace. There are just three sides on URM machines, just two on Turing machines. The reader is left to work out the motivation for the official definition of a primitive recursive function from the examples that actually follow that definition. Again, the move from primitive recursive to partial recursive functions is done at pace (and after just a page, we immediately meet an Ackermann-style function as an example of an intuitively computable and total recursive but not primitive recursive function). None of the ideas thus far are hard, of course: but there are some quite excellent, rather less breathless and hence more illuminating, alternative treatments available.

Similar remarks apply to the next two chapters (so this going to be a repetitive theme!). Ch. 10 is on universal machines and undecidability. It is shown that there is a universal computable functions $\Psi\colon \mathbb{N}^2 \to \mathbb{N}$ such that if $\varphi_n(x)$ is $\Psi(n, x)$, then $\varphi_0, \varphi_1, \varphi_2, \ldots$ is an enumeration of all the computable (one-place) partial functions. We meet Kleene's T-predicate, the s–m–n theorem, then the unsolvability of the halting problem leading to the undecidability of first-order validity. And next it is on to Rice's theorem and the (second) Recursion Theorem – with all this in the space of just 11 pages! Really?

Then Ch. 11, only a couple of pages longer, discusses computably enumerable sets, many-one reducibility, computably inseparable sets, the arithmetical hierarchy, Kolmogorov randomness and more. To be frank, I see nothing to be gained by rushing through at this pace. However mathematically ept the beginner, they will assuredly get a better conceptual understanding of what is going on – especially if engaged in solo self-study – by tackling, e.g., the more expansive early chapters of Cutland's classic text instead of these three rushed chapters by Marker.

Now, Chs 9 to 11 may fly by at unhelpful speed, but the topics covered there are, we can readily agree, entry-level. By contrast, Ch. 12 on Turing Reducibility rachets up the level of sophistication significantly – it is still only 14 pages, yet we get as far as Friedburg-Muchnik (which e.g. Cutland says is beyond the scope of his book, but which is proved by Cooper in his more advanced text but not starting until his p. 238) and the Low Basis Theorem (Cooper p. 330). Now, assuming you come primed with a strong enough understanding of basic computability theory, you could perhaps usefully tackle this chapter (these upper-level topics are of course intrinsically interesting). But again my sense is that the slower presentation of Friedburg-Muchnik in Cooper (say) is significantly more likely to engender a deeper understanding of the priority method used in the proof.

So, a summary verdict on Part III: too much is done too quickly for a first encounter with this material (there are other treatments, including ones still primarily aimed at graduate mathematicians, which will be more inviting). Of course, Part III could be useful revision/consolidation material for enthusiasts who like Marker's brusque style; though, by my lights, there are more attractive alternatives for that too.

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Part IV of Marker's book, 'Arithmetic and Incompleteness' is the longest, at over 100 pages, but I'll be briefer.

The first chapter, Ch. 13 on the incompleteness theorems, is reasonably accessible, though for various reasons it wouldn't be my recommendation for a place to start on the topic; still, this could well provide useful follow-up reading for beginners.

Ch. 14 is on Hilbert's 10th problem. We don't get quite a full proof of the MRDP theorem, with all the dots joined up; but this is pretty clearly done, I think, and so (without too many tears) you'll get a decent sense of what is going on. However, the nice book in the AMS Student Mathematical Library by Murty and Fodden is still clearer, more inviting, and indeed more complete: I'd probably recommend reading the appropriate sections of that instead.

Ch. 15 is titled 'Peano Arithmetic and ϵ_0'. This long chapter aims at a proof of the Kirby-Paris theorem that Goodstein's Theorem is unprovable in PA. As Marker himself clearly acknowledges with thanks, the line of argument follows closely an unpublished piece by Henry Towsner. I think you'll want to read Marker's chapter and Towsner's piece in tandem – Marker is clearer, e.g., about the Hardy hierarchy of fast-growing functions, Towsner is perhaps clearer about what's going on with the Schütte-style infinitary deduction system for arithmetic on which the overall proof turns. This two-pronged approach should then work well, and I think this is the chapter of Marker's book that I found the most helpful addition to the literature.

The shorter final Ch. 16 is titled 'Models of Arithmetic and Independence Results'. After a section on provably total functions of PA, the chapter dashes on to establish the unprovability in PA of the Paris-Harrington Principle in Ramsey Theory. So there is some speedy setting-up of context, and then a dense proof. Then the discussion rushes on to a number of other results in the model theory of PA (Gaifman's Splitting Theorem, Bounded Recursive Saturation, Tennenbaum's Theorem). We are back, then, to topics tackled at great pace. Almost anyone who wants to understand this material will be much better off working through Kaye's approachable – indeed, one might say, particularly inviting – book Models of Peano Arithmetic.

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Summary verdict Overall I'm disappointed. Marker's book has its moments, but it too often provides a bumpy, breathlessly fast, ride – so it is not so much the promised inviting introduction as a book that can be mined for supplementary/more advanced reading on some of its topics.

September 2024

30 Westerståhl, 2024

Dag Westerståhl's *Foundations of Logic: Completeness, Incompleteness, Computability* (CSLI 2024, pp. 451) is, by some margin, the most attractively written book at its level that I have newly encountered for some years.

The book is based on lecture courses give at Stockholm University and latterly at Tsinghua University, Beijing, and evidently reflects long teaching experience. Especially in the first two Parts of the book, the expositional choices are very well-judged, and the balance between motivational chat and worked-through formal details seems just right to me. Many student readers should find this book quite excellent for self-study.

In just a little more detail, Part I ('Background', 78 pp.), introduces propositional and FOL languages, and the idea of interpretations and semantic consequences, and then gives both a natural deduction proof system (Gentzen-style) and a Hilbert proof system. Part II ('Completeness', 80 pp.) gives soundness and completeness theorems for propositional logic and FOL, and then there is a chapter on the L-S theorems, compactness, and a smidgin of model theory. Part III ('Incompleteness', 134 pp.) discusses primitive recursive functions and their representability, Peano arithmetic, arithmetization, and Gödel's Theorems. Part IV ('Computability', 114 pp.) adds chapters on decidability, undecidability and a modest amount more computability theory. There is an Appendix (34 pp.) on sets and functions, etc., for those that need it.

So this is a very substantial book. But Parts I and II in particular strike me as admirably well-paced, to my mind neither over-packed with detail, nor tediously slow-moving.

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A little more detail on Parts I and II. After an inviting, reader-friendly, Introduction (Ch. 1), Part I has two substantial chapters, the first one (Ch. 2) introducing the syntax and semantics of propositional logic and then of FOL. This is done conventionally, in that we get the usual Tarski-style story, with the semantics done via assignments of objects to all variables at once in the usual way, interpreting wffs with free variables. So, in the usual way, we need slightly messy stories about allowable substitutions avoiding variable-capture, slightly messy notation for handling assignments to variables, etc. However, this is all expounded particularly clearly, with motivations at various points very well explained. (There's perhaps just a hint in §3.2 that we could do things differently, in the way I myself would prefer, using a supply of constants-as-parameters, rather than making variables do double duty – and then we can assign parameters temporary interpretations one at a time, just on an as-needed basis. But I can of course see the virtues of sticking to the conventional Tarskian line.)

Then Ch. 3 introduces two proof systems, Gentzen-style natural deduction and a minimalist, two-connective, Hilbert-style system. I think I would probably have spent just two or three more pages giving more examples of Gentzen-style proofs, showing tactics for building up proof-trees in natural ways (to re-inforce the advertised claim that this really is a *natural* deduction system). But still, this is again all very well done, with possible stumbling blocks – e.g. the use of vacuous discharge – nicely elucidated. The chapter finishes particularly helpfully with an outline proof that the two proof systems march in step, warranting the same deductions.

Part II begins with a short chapter (Ch. 4) proving the completeness of the Hilbert proof system for propositional logic, using maximal consistent sets and Lindenbaum's Lemma. As in previous (and future!) chapters, there are excellent exercises, including in this case ones exploring another completeness proof and presenting the interpolation lemma for PL. Then, closely following that chapter's overall structure, we get a short chapter (Ch.5) proving the completeness theorem for FOL using Henkin's method – very sensibly doing the proof for FOL without identity first, before in the final section of chapter complicating the story to handle identity. I wonder if that last section is just a bit denser than it needs to be – but overall, again, this seems quite excellently well done.

The rest of Part II concludes with a significantly longer chapter (Ch. 6) titled, simply, 'Model theory'. The first eight sections introduce a handful of predictable entry-level topics: more ideas about structures, about isomorphisms between structures, notions of definability, the compactness theorem (as a corollary of completeness) and some of its applications, and the Löwenheim-Skolem theorems. The remaining sections are more challenging, and could well be skipped on a first reading. §6.9 reflects Westerståhl's interest in generalized quantifiers and quantifiers in natural language. §6.10 is on Ehrenfeucht–Fraïssé games. §6.11, ambitiously, is on Lindström's Theorem.

Suppose we stop before those last three sections (though take in the still-relevant end-of chapter exercises). Then what we have before us is a book-within-a-book – Westerståhl's 150 page introduction to FOL. There are, of course, oodles of books on the same material (and maybe covering a lot more). However, for one reason or another, many respectable texts on FOL can be difficult to recommend very warmly for self-study (over-doing the "rigour", under-doing the motivational chat, choosing a horrible deductive proof system, etc., etc.).

Against this background, the first two parts of *Foundations of Logic* do seem to me to stand out as providing a particularly attractive option.

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A little more detail on Parts III and IV. Again, there are many more books which introduce formal arithmetic, incompleteness, and some entry-level computability theory. But this time, we are already rather generously supplied with a number of particularly accessible and enjoyable options taking varying paths through the material; and I'm not sure that Parts III and IV of Westerståhl's book trump the alternatives. Which is not to deny that the book continues to be very attractively written, well-organized with a lot of signposting. In particular, Westerståhl does a very good job in signalling what are the Big Ideas, and what is the hack work required to confirm that e.g. primitive recursive functions are representable in PA, or that syntax is sufficiently arithmetizable, etc. It then becomes a judgment call just how many of the tedious under-the-bonnet details you really need to go into.

In fact, the particular path to the incompleteness results that Westerståhl takes closely resembles the one in my own *IGT* (which gets a friendly mention in his preface). But at various points he goes just a little more into the nitty-gritty detail than I do – more, obviously enough, than I judged was necessary. And, despite the signposting, I can well imagine some student readers occasionally losing orientation. So (what a surprise!) if you want to follow our path to Gödel's Theorems – and there are alternatives – I'd still recommend starting with *IGT* for self-study. Westerståhl's Parts III and IV would then make for quite excellent follow-up reading to consolidate understanding.

Part III ('Incompleteness') comprises six chapters. Ch. 7 is a very nice chapter, outlining what is to come – and indeed could be read stand-alone for preliminary orientation on incompleteness and computability, whatever you go on to read as your main text(s). Ch. 8 is about primitive recursive functions. Ch. 9 introduces first-order Peano Arithmetic. Ch. 10 establishes the representability of p.r. functions. Ch. 11 is on the arithmetization of syntax. So far so good, if sometimes unnecessarily detailed? Then, the target of this Part, Ch. 12 is a distinctly action-packed, indeed over-busy, forty-page chapter on incompleteness (*IGT* takes over twice as many pages to cover much of the same material as in this chapter, and I really think is all the better for it as a text for self-study by beginners).

Part IV ('Computability') has three chapters. Ch. 13 is on Turing machines, recursive functions and decidability. Ch. 14 is on undecidability of extensions of PA, of FOL, of the halting problem (so these two chapters correspond to much of final half-dozen chapters of *IGT*). And then Ch. 15 dips its toes into computability theory (occasionally going rather beyond anything in *IGT*, e.g. in introducing the s-m-n theorem, creative sets, etc.). All good stuff, and I should certainly note again that the end-of-chapter Exercises – as earlier in the book – continue to be really excellent.

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Summary verdict The first two Parts of this book take their place as among the very top recommendations for self-study on FOL.

The second two Parts wouldn't be my own recommended first entry-point for self-study on arithmetic/incompleteness/computability. However, those who would prefer a somewhat brisker trek through this material along much the same path as *IGT* with a few extra sights along the way will again find Westerståhl's book a very admirable guide.

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